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TEST-SCORE SEMANTICS FOR NATURAL LANGUAGES AND MEANING REPRESENTATION VIA PRUF

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Abstract

In a sharp departure from conventional approaches to the problem of meaning representation in natural languages, test-score semantics is based on the premise that almost everything that relates to natural languages is a matter of degree. Thus, in test-score semantics, predicates, propositions and other types of linguistic entities are treated as collections of elastic constraints on a set of objects or relations in a universe of discourse. Viewed in this perspective, the meaning of a linguistic entity may be defined by: (a) identifying the constraints which are implicit or explicit in the entity in question; (b) describing the tests that must be performed to ascertain the degree to which each constraint is satisfied; and (c) specifying the manner in which the degrees in question or, equivalently, the partial test scores are to be aggregated to yield an overall test score. In general, the overall test score is a vector whose components are numbers in the unit interval or possibility/probability distributions over this interval.

The first step in the representation of the meaning of a given proposition involves the construction of a relational database in which the meaning of constituent relations and their attributes is assumed to be known. The choice of the database affects the explanatory effectiveness of the translation process and is governed by the knowledge profile of the intended user of the translation. The test procedure--which is regarded as the representation of the meaning of the proposition--acts on the database and returns an overall test score which is interpreted as the compatibility of p with the database.

Test-score semantics is sufficiently general to allow the translation into PRUF of almost any proposition in a natural language. However, the price of generality is the difficulty of writing a program which could represent the meaning of a given proposition without recourse to human assistance.

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Test-Score Semantics for Natural Languages and
Meaning Representation Via PRUF

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1. Introduction

There are some philosophers of language who believe, as Montague did [75], that the construction of a rigorous mathematical theory of natural languages is an attainable objective. An opposing point of view, which is articulated in the present paper, is that no mathematical theory based on two-valued logic is capable of mirroring the elasticity, ambiguity and context-dependence which set natural languages so far apart from the synthetic models associated with formal syntax and set-theoretic semantics.

To Professor Max Black

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The basis for our contention is that almost everything associated with natural languages is a matter of degree. This applies, in particular, to the issue of grammaticality and, even more so, to the notion of meaning. Thus, any logical system in which there are no gradations of truth and membership is ipso facto unsuitable as a framework for a comprehensive theory of natural languages and, especially, for the representation of meaning, knowledge and strength of belief.

As an alternative to the approaches based on two-valued logic, we have proposed in [139] a meaning-representation language PRUF¹ in which an essential use is made of what may be described as possibility theory [138], [140]. This theory--which is distinct from the bivalent theories of possibility related to modal logic and possible-world semantics [47], [92] --is based on the concept of a possibility distribution, which in turn is analogous to, and yet distinct from, that of a probability distribution. In effect, the basic idea underlying PRUF is that the concept of a possibility distribution provides a natural mechanism for the representation of much of the imprecision and lack of specificity which is intrinsic in communication between humans.

As will be seen in the sequel, the expressive power of PRUF is substantially greater than that of predicate calculus, Montague grammar, semantic networks, conceptual dependency, and other types of meaning-representation systems that are currently in use [27], [68], [70], [101].

¹Actually PRUF is not just a language but a meaning-representation system, which includes a language as one of its components.

In particular, PRUF makes it possible to represent the meaning of propositions which contain (a) fuzzy quantifiers, e.g., many, most, few, almost all, etc.; (b) modifiers such as very, more or less, quite, rather, extremely, etc.; and (c) fuzzy qualifiers such as quite true, not very likely, almost impossible, etc. However, the price of being able to translate almost any proposition in a natural language into PRUF is the difficulty of establishing a homomorphic connection between syntax and semantics -- as is done in Montague grammar for fragments of English, and in Knuth semantics and attributed grammars for programming languages [57]. What this implies is that, although it is relatively easy to teach a human subject to translate from a natural language into PRUF, it would be very hard to write a program that could perform similarly without human assistance or intervention.

The semantics underlying PRUF is what we shall refer to as test-score semantics -- a semantics in which the concept of aggregation of test scores plays a central role. Test-score semantics subsumes most of the semantical systems which have been proposed for natural languages and, in particular, includes as limiting cases both truth-conditional and possible-world semantics [70], [68], [18].

The basic idea behind test-score semantics may be summarized as follows. An entity in linguistic discourse, e.g., a predicate, a proposition, a question or a command, has, in general, the effect of inducing elastic constraints on a set of objects or relations in a universe of discourse. The meaning of such an entity, then, may be defined by (a) identifying the constraints which are induced by the entity; (b) describing the tests that must be performed to ascertain the degree to which each constraint is satisfied; and (c) specifying

the manner in which the degrees in question or, equivalently, the partial test scores are to be aggregated to yield an overall test score. Viewed in this perspective, then, the meaning of a linguistic entity in a natural language may be identified with the testing of elastic constraints which are implicit or explicit in the entity in question.

We shall begin our exposition of PRUF and test-score semantics with a brief review of some of the basic notions in possibility theory which will be needed in later sections. A more detailed exposition of possibility theory may be found in [138], [140], [83], [45], [130] and [26].²

²Some of the definitions and examples in Section 2 are drawn from [137] and [140].

2. The Concept of Possibility Distribution

Informally, let X be a variable which takes values in a set U . Then, the possibility distribution of X , denoted by Π_X , is the fuzzy set of possible values of X , with the understanding that possibility is a matter of degree. Thus, if u is a possible value of X , we shall write

$$\text{Poss}\{X=u\} = a \quad (2.1)$$

to indicate that the possibility that X can take u as its value is a , where a is a number in the interval $[0,1]$.

The function $\pi_X : U \rightarrow [0,1]$ which associates with each $u \in U$ the possibility that X can take u as its value is called the possibility distribution function. Thus

$$\text{Poss}\{X=u\} = \pi_X(u), u \in U, \quad (2.2)$$

where U is the domain of X . In effect, the possibility distribution function π_X is the membership function of the possibility distribution Π_X .

In general, a possibility distribution may be induced by a physical constraint or, alternatively, may be epistemic in origin. To illustrate the difference, let X be the number of passengers that can be carried in Suppes' car, which is a five passenger Mercedes. In this case, by identifying $\pi_X(u)$ with the degree of ease with which u passengers can be put in Suppes' car, the tabulation of π_X may assume the following form:

| | | | | | | | | | | |
|---------|---|---|---|---|---|-----|-----|-----|---|----|
| X | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| π_X | 1 | 1 | 1 | 1 | 1 | 0.8 | 0.6 | 0.2 | 0 | 0 |

in which an entry such as (7,0.6) signifies that, by some explicit or implicit criterion, the degree of ease with which 7 passengers can be carried in Suppes' car is 0.6.

In the above example, the possibility distribution of X is induced by a physical constraint on the number of passengers that can be carried in Suppes' car. To illustrate the case where the possibility distribution of X is epistemic in origin, i.e., reflects the state of knowledge about X, let X be Suppes' height and let the information about Suppes' height be conveyed by the proposition

$$p \triangleq \text{Suppes is tall}, \quad (2.3)$$

where tall is the label of a specified fuzzy subset of the interval [0, 250 cm] which is characterized by its membership function μ_{TALL} , with $\mu_{\text{TALL}}(u)$ representing the degree to which a person whose height is u cm is tall in a specified context.

The connection between the variable $X \triangleq \text{Height (Suppes)}$, the proposition $p \triangleq \text{Suppes is tall}$, and the fuzzy set TALL^3 is provided by the so-called possibility postulate of possibility theory [137], [139], which for the example under consideration implies that, in the absence of any information about X other than that supplied by p, the possibility that $X=u$ is numerically equal to the grade of membership of u in TALL.

³We use uppercase letters to represent fuzzy sets and fuzzy relations.

Thus

$$\text{Poss}\{X=u\} = \pi_X(u) = \mu_{\text{TALL}}(u), \quad u \in [0, 250] \quad (2.4)$$

or, equivalently,

$$\Pi_{\text{Height(Supes)}} = \text{TALL}, \quad (2.5)$$

where (2.5) is referred to as the possibility assignment equation. In summary, we shall say that p translates into the possibility assignment equation (2.5), i.e.,

$$\text{Supes is tall} \rightarrow \Pi_{\text{Height(Supes)}} = \text{TALL}, \quad (2.6)$$

where the arrow \rightarrow should be read as "translates into."

More generally, a central idea in PRUF is that any proposition in a natural language that may be put into the canonical form

$$p \triangleq N \text{ is } F, \quad (2.7)$$

where N is the name of an object, a variable or a proposition, may be interpreted as a characterization of the joint possibility distribution of a collection of variables X_1, \dots, X_n which are implicit or explicit in p . Thus, in symbols, $N \text{ is } F$ translates into

$$N \text{ is } F \rightarrow \Pi(X_1, \dots, X_n) = F. \quad (2.8)$$

The variables X_1, \dots, X_n which are constrained by the possibility assignment equation will be referred to as the base variables of p .

Example. Consider the proposition

Nils has a large office

which may be expressed as

$p \triangleq$ Nils' office is large .

In this case, the implicit base variables are:

$X_1 = \text{Length}(\text{Office}(\text{Nils}))$

$X_2 = \text{Width}(\text{Office}(\text{Nils}))$

and the possibility assignment equation assumes the form

$$\Pi(\text{Length}(\text{Office}(\text{Nils}), \text{Width}(\text{Office}(\text{Nils}))) = \text{LARGE} , \quad (2.9)$$

where LARGE is a fuzzy set or, equivalently, a fuzzy binary relation in the product space LENGTH \times WIDTH. Thus, (2.9) implies that

$$\text{Poss}\{\text{Length}(\text{Office}(\text{Nils})) = u, \text{Width}(\text{Office}(\text{Nils})) = v\} = \mu_{\text{LARGE}}(u,v) ,$$

where $\mu_{\text{LARGE}}(u,v)$ is the degree to which an office which is u long and v wide is defined to be large in a specified context.

What is the difference between probability and possibility? As the above examples indicate, the concept of possibility is an abstraction of our intuitive perception of ease of attainment or the degree of compatibility, whereas the concept of probability is rooted in the perception of likelihood, frequency, proportion or strength of belief. Furthermore, as we shall see in Section 3, the rules governing the manipulation of possibilities are distinct from those which apply to probabilities.

An important aspect of the connection between probabilities and possibilities relates to the fact that, in principle, they are

independent characterizations of uncertainty in the sense that, from the knowledge of the possibility distribution of a variable X , we cannot deduce its probability distribution, and vice-versa. For example, from the knowledge of the possibility distribution of the number of passengers in Suppes' car we cannot deduce its probability distribution; nor can we deduce the possibility distribution from the probability distribution of the number of passengers. In general, however, we can make a vague assertion to the effect that, if the possibility that $X=u$ is small, then it is likely that the probability that $X=u$ is also small. However, from this it does not follow that high possibility implies high probability, as is reflected in the commonly used statements of the form "It is possible but not probable that...."

If the translation of a proposition p in a natural language is taken to be a possibility assignment equation as represented (2.8), then a question that naturally arises is: How can the base variables X_1, \dots, X_n and their joint possibility distribution $\Pi_{(X_1, \dots, X_n)}$ be determined from p ?

At this juncture in the development of PRUF, we do not have an algorithm for identifying the base variables in a given proposition. However, experience has shown that it is not difficult for a human subject to acquire a facility for translating any proposition within a broad class of propositions into a possibility assignment equation. What is difficult, as was alluded to already, is mechanizing this process completely, so that the translation represented by (2.8) could be accomplished without any human assistance.

In PRUF, the translation of a proposition may be either focused⁴ or unfocused, with the focused translation leading, in general, to a possibility assignment equation. The unfocused translation--of which the focused translation is a special case--is based on test-score semantics and has the form of (i) a collection of tests which are performed on the database induced by the proposition; and (ii) a set of rules for aggregating the partial test scores into a overall test score which represents the compatibility of the given proposition with the database. In what follows, we shall present a condensed exposition of test-score semantics and illustrate its use in PRUF by a number of examples.

⁴The concept of focusing in test-score semantics differs from that introduced by B. Grosz [37] in the context of partitioned semantic networks.

3. Test-Score Semantics: Nature of Tests

To simplify our exposition of test-score semantics, it will be convenient to focus our attention on the representation of the meaning of propositions, with the understanding that the basic ideas underlying test-score semantics are equally applicable to predicates, questions, commands and most other types of linguistic entities.

As will be seen in the sequel, the conceptual framework of test-score semantics is rooted -- like that of truth-conditional semantics -- in our intuitive perception of meaning as a collection of criteria for relating a linguistic entity to its designation. More specifically, suppose that we wish to test whether or not a human subject, H, understands the meaning of a proposition p, e.g., $p \triangleq$ Laura is dancing with Irwin. A natural way of doing this would be to present H with a variety of scenes (or worlds) depicting a joint activity of Laura and Irwin, and ask H to indicate, for each scene or world W, the degree, $c(W)$, to which W corresponds to or is compatible with H's perception of the meaning of p. If H can do this correctly for each W, then we may conclude that H understands the meaning of p. And, more importantly, if H can articulate the tests which H performs on W to arrive at $c(W)$, then H not only understands what p means ostensively, but can also precisiate the meaning of p by a concretization of the test procedure.

In truth-conditional and possible-world semantics, the degree of compatibility, $c(W)$, is allowed to have one of two possible values: {true, false} or, equivalently, {pass, fail}. By contrast, in test-score semantics, $c(W)$ can be any point in a linear or partially ordered set--which for simplicity is usually taken to be the unit interval

[0,1]. Furthermore, $c(W)$ is also allowed to be a probability or possibility distribution over the unit interval or, more generally, a composition of probability and possibility distributions.

Instead of dealing with scenes or worlds directly, it is simpler and more effective to deal with their characterizations in the form of state descriptions (Carnap [17]) or, equivalently, as databases. In essence, then, we assume that H is presented, on the one hand, with a proposition p and, on the other, with a database D , and that H performs a test, T , on D which yields a test score, τ . In symbols,

$$\begin{aligned}\tau &= T(D) && (3.1) \\ &= \text{Comp}(p,D),\end{aligned}$$

where the test T may be viewed as a representation of the meaning of p , and its test score, τ , as a measure of the compatibility of p and D . Furthermore, viewed from the perspective of truth-conditional semantics, τ may be interpreted as the truth-value of p given D , i.e.,

$$\tau = \text{Tr}\{p|D\} . \quad (3.2)$$

Alternatively, p may be interpreted -- in the spirit of possible-world semantics -- as the possibility of D given p , i.e.,

$$\tau = \text{Poss}\{D|p\} . \quad (3.3)$$

In general, a test, T , is composed of a number of constituent tests, T_1, \dots, T_n , and the overall test score, τ , is the result of aggregation of constituent test scores τ_1, \dots, τ_n , where τ_i , $i = 1, \dots, n$, is the test score associated with T_i . In test-score semantics, the

process of aggregation need not be carried to the extreme of yielding a single test score, i.e., a number in the interval $[0,1]$. Thus, more generally, the aggregated test score τ may be a vector, $\tau = (\tau_\alpha, \dots, \tau_\gamma)$, in which each of the components is a number in the interval $[0,1]$ or a probability/possibility distribution over the unit interval. In particular, the analysis of presuppositions requires the use of vector test scores to differentiate the results of tests performed on presuppositions from those performed on other constituents of the proposition under analysis.⁵

Notational preliminaries

We shall assume that a database consists of a collection of relations, each of which is represented by (a) its relational frame, i.e., the name of the relation and the names of variables (columns); and (b) the data, i.e., the entries in the table. For example, the relational frame of the relation named POPULATION:

| POPULATION | Name | Age | Height (cm) |
|------------|---------|-----|-------------|
| | Minker | 38 | 170 |
| | Rieger | 36 | 182 |
| | Sanchez | 36 | 175 |

may be expressed as POPULATION || Name | Age | Height

or equivalently as POPULATION [Name; Age; Height].

⁵The idea of a vector truth-value was suggested earlier by the author (see [60]). The concept of a vector test score as defined in this paper provides a more general framework for the analysis of presuppositions than that of two-dimensional languages (Herzberger [43], McCawley [68], Bergmann [11]).

Generally, we shall be dealing with fuzzy relations of the form:⁶

| | | | | | |
|---|----------|----------|-----|----------|---------|
| R | X1 | X2 | ... | Xn | μ |
| | . | . | . | . | . |
| | r_{t1} | r_{t2} | . | r_{tn} | μ_t |
| | . | . | . | . | . |

in which r_{tk} , $k = 1, \dots, n$, is the entry in t^{th} row and column X_k , and μ_t is the grade of membership of the n-tuple $r_t \triangleq r_{t1} \dots r_{tn}$ (or (r_{t1}, \dots, r_{tn})) in the fuzzy relation R. For example, in the relation BIG

| | | | |
|-----|-------------|------------|-------|
| BIG | Length (cm) | Width (cm) | μ |
| | 35 | 28 | 0.7 |
| | 45 | 39 | 0.9 |

the entries in the second row signify that an object which is 45 cm long and 39 cm wide is defined to be big to the degree 0.9. In effect, R may be viewed as an elastic constraint on the n-ary variable $X \triangleq (X_1, \dots, X_n)$, with $\mu_t \triangleq \mu_R(r_{t1}, \dots, r_{tn})$ representing the degree to which an n-tuple (r_{t1}, \dots, r_{tn}) of values of X_1, \dots, X_n satisfies the constraint in question. When it is desirable to place in evidence that R is a constraint on X, we shall express R as R_X or, more explicitly, as $R(X_1, \dots, X_n)$.

⁶Here and elsewhere in the paper, the subscripts on variables are raised for typographical convenience.

Let $s \triangleq (i_1, \dots, i_k)$ be a subsequence of the index sequence $(1, \dots, n)$, and let s' denote the complementary subsequence $s' \triangleq (j_1, \dots, j_m)$ (e.g., for $n = 5$, $s = (1, 3, 4)$ and $s' = (2, 5)$). In terms of such sequences, a k -tuple of the form $(r_{i_1}, \dots, r_{i_k})$, where r is an arbitrary symbol, may be expressed in an abbreviated form as $r_{(s)}$ (or $r(s)$). Expressed in this notation, the variable $X_{(s)} \triangleq (X_{i_1}, \dots, X_{i_k})$ will be referred to as a k -ary subvariable of $X \triangleq (X_1, \dots, X_n)$, with $X_{(s')} \triangleq (X_{j_1}, \dots, X_{j_m})$ being a subvariable complementary to $X_{(s)}$.

Projection

An operation which plays a basic role in the manipulation of fuzzy relations and possibility distributions is that of projection. Specifically, assume that X_i , $i = 1, \dots, n$, takes values in the universe of discourse $U_i(X_i) \triangleq U_i$. Then, the projection of R on the domain $U(s) \triangleq U_{i_1} \times \dots \times U_{i_k}$ is expressed as:⁷

$$\begin{aligned} \text{Proj}_{U(s)} R &\triangleq X_{i_1} \times \dots \times X_{i_k} R & (3.4) \\ &\triangleq X_{(s)} R . \end{aligned}$$

The grade of membership of a k -tuple $u(s) \triangleq (u_{i_1}, \dots, u_{i_k})$ in $X_{(s)} R$ is defined by

$$\mu_{X_{(s)} R}(u_{i_1}, \dots, u_{i_k}) \triangleq \sup_{u(s')} \mu_R(u_1, \dots, u_n), \quad (3.5)$$

⁷This notation for projections is patterned after the notation employed in the query language SQUARE [14].

where the notation $\sup_{u(s')}$ signifies that the supremum is taken over the domain of the complementary subvariable $u(s') \triangleq (u_{j_1}, \dots, u_{j_m})$. Stated more simply, the operation of projection on $U_{i_1} \times \dots \times U_{i_k}$ has the effect of deleting the components u_{j_1}, \dots, u_{j_m} in the n -tuple (u_1, \dots, u_n) and associating with the resulting k -tuple the highest grade of membership among all n -tuples in R in which $x_{i_1} = u_{i_1}, \dots, x_{i_k} = u_{i_k}$. To illustrate, if R is given as

| R | X1 | X2 | X3 | μ |
|---|----|----|----|-------|
| | a | a | a | 0.7 |
| | a | a | b | 0.8 |
| | a | b | a | 0.2 |
| | a | b | b | 1 |
| | b | a | a | 0.4 |
| | b | b | a | 0.6 |

then the projections of R on $U_1 \times U_2$ and U_3 are:

| $X1 \times X2$ | X1 | X2 | μ |
|----------------|----|----|-------|
| | a | a | 0.8 |
| | a | b | 1 |
| | b | a | 0.4 |
| | b | b | 0.6 |

| $X3$ | X3 | μ |
|------|----|-------|
| | a | 0.7 |
| | b | 1 |

Note that, from the definition of projection (3.5), it follows at once that

$$X(\alpha)^R \equiv X(\alpha)(X(\beta)^R)$$

provided every variable that is in the index sequence α is also in the index sequence β .

Particularization

A basic operation on fuzzy relations which plays an important role in test-score semantics is that of particularization.⁸ More specifically, assume that R is an n -ary relation which represents a constraint on an n -ary variable $X \triangleq (X_1, \dots, X_n)$. Now suppose that we impose an additional constraint, G , on a subvariable of X , say $X(s) \triangleq (X_{i_1}, \dots, X_{i_k})$. Then, the additional constraint on X may be viewed as a particularization of R , expressed in symbols as

$$R[X(s) \text{ is } G] \tag{3.6}$$

or, equivalently, in virtue of (2.8), as

$$R[\Pi_{X(s)} = G] , \tag{3.7}$$

where $\Pi_{X(s)}$ is the possibility distribution of the subvariable $X(s)$.

Remark. In some cases it is necessary to differentiate between two different interpretations of propositions of the form "X is F." In what we shall refer to as the possibilistic (or disjunctive) interpretation, the translation of "X is F" is $\Pi_X = F$. On the other hand, in the conjunctive interpretation, "X is F" is interpreted as $X = F$, which in turn means that if F is a fuzzy set expressed as⁹

⁸In the case of nonfuzzy relations, particularization reduces to what is commonly referred to as restriction or selection.

⁹In this notation, μ_i/u_i signifies that μ_i is the grade of membership of u_i in F , and $+$ denotes the union rather than the arithmetic sum.

$$F = \mu_1/u_1 + \dots + \mu_N/u_N \quad (3.8)$$

or, equivalently, as

$$F = \sum_i \mu_i/u_i \quad (3.9)$$

then each u_i is a value of X to the degree μ_i .

On occasion, to differentiate between the disjunctive and conjunctive interpretations, we shall employ the more explicit notation

$$X = \text{dis}(F) \quad \text{in place of } \Pi_X = F \quad (3.10)$$

and

$$X = \text{con}(F) \quad \text{in place of } X = G \quad (3.11)$$

with the understanding that, unless stated to the contrary, "X is F" should be interpreted as $X = \text{dis}(F)$. An example illustrating the difference between disjunctive and conjunctive interpretations is given in [139].

To give a concrete meaning to (3.7), it is convenient to employ the concept of a row test. Specifically, let $r_t = (u_{1t}, \dots, u_{nt}, \mu_t)$ be the t^{th} row of R , where $u_{1t}, \dots, u_{nt}, \mu_t$ are the values of X_1, \dots, X_n, μ respectively. Furthermore, let μ_G be the membership function of the fuzzy set G which appears in (3.7), and let v_t be the grade of membership of $r_t(s)$ in G , i.e.,

$$v_t = \mu_G(r_t(s)). \quad (3.12)$$

In terms of the parameters just defined, the row test in question may be described as follows. First, determine the degree to which $r_t(s)$ satisfies the particularizing constraint "X(s) is G" by

setting the test score equal to v_t ; and second, combine μ_t and v_t by employing the min operator \wedge , yielding the aggregated test score

$$\tau_t = \mu_t \wedge v_t . \quad (3.13)$$

Remark. The aggregation operator \wedge (min) should be viewed as a default choice when no alternative is specified. When an aggregation operator * other than min is specified (e.g., arithmetic mean, product, geometric mean, etc.¹⁰) the expression for τ_t becomes

$$\tau_t = \mu_t * v_t . \quad (3.14)$$

Once the aggregated test score is found for each row in R, the particularized relation $R[\Pi_{X(s)} = G]$ is readily constructed by replacing μ_t in r_t by τ_t , resulting in the modified $(n+1)$ -tuple

$$r_t^* = (u_{1t}, \dots, u_{nt}, \tau_t) ,$$

which represents the t^{th} row of $R^* \triangleq R[\Pi_{X(s)} = G]$ and in which τ_t is expressed by (3.13) or, more generally, by (3.14).

A row test is compartmentalized when more than one particularizing constraint is involved, as in:

$$R[X(s) \text{ is } F; X(v) \text{ is } G] , \quad (3.15)$$

¹⁰The closely related issue of various ways in which operations on fuzzy sets may be defined has received considerable attention in the literature. See, in particular, [25], [56], [131] and [142].

where $X(s)$ and $X(v)$ are not necessarily disjoint subvariables of X . In this case, let $v_t^{(F)}$ and $v_t^{(G)}$ be the test scores associated with the row tests " $X(s)$ is F " and " $X(v)$ is G ," respectively. Then, using the default definition of the aggregation operator $*$, the aggregated test score for r_t may be expressed as

$$\tau_t = \mu_t \wedge v_t^{(F)} \wedge v_t^{(G)} . \quad (3.16)$$

Example. Consider a relation R defined by the table

| R | X1 | X2 | X3 | μ |
|---|----|----|----|-------|
| | a | a | a | 1 |
| | a | b | a | 0.8 |
| | b | a | a | 0.6 |
| | b | b | b | 0.3 |

which is particularized by the constraints

$$(X_1, X_2) \text{ is } F$$

$$(X_2, X_3) \text{ is } G ,$$

where F and G are defined by

$$F = 1/(a,a) + 0.6/(b,a) + 0.2/(b,b)$$

$$G = 0.3/(a,a) + .9/(a,b) + 0.7/(b,a) + 0.4/(b,b)$$

with the understanding that a term such as $0.6/(b,a)$ in F signifies that the grade of membership of the tuple (b,a) in F is 0.6.

Applying the compartmentalized row test to the rows of R , we obtain successively

$$v_1(F) = 1 \quad ; \quad v_1(G) = 0.3; \quad \tau_1 = 1 \wedge 1 \wedge 0.3 = 0.3$$

$$v_2(F) = 0 \quad ; \quad v_2(G) = 0.7; \quad \tau_2 = 0.8 \wedge 0 \wedge 0.7 = 0$$

$$v_3(F) = 0.6; \quad v_3(G) = 0.3; \quad \tau_3 = 0.6 \wedge 0.6 \wedge 0.3 = 0.3$$

$$v_4(F) = 0.2; \quad v_4(G) = 0.4; \quad \tau_4 = 0.3 \wedge 0.2 \wedge 0.4 = 0.2$$

and hence $R^* \triangleq R[\Pi_{(X_1, X_2)}=F; \Pi_{(X_2, X_3)}=G]$ is given by the table

| R^* | X1 | X2 | X3 | τ |
|-------|----|----|----|--------|
| | a | a | a | 0.3 |
| | b | a | a | 0.3 |
| | b | b | b | 0.2 |

In the foregoing discussion, we have discussed the concepts of projection and particularization in the context of operations on relations. Inasmuch as a possibility distribution is a relation which acts as a disjunctive constraint on the values of a variable, the operations of projection and particularization apply equally well to possibility distributions. For example, we may write

$$\Pi_1 = \Pi[\Pi_{(X_1, X_2)}=F; \Pi_{(X_2, X_4)}=G] \quad (3.17)$$

to indicate that Π_1 is a possibility distribution which results from particularizing the possibility distribution Π with the constraints " (X_1, X_2) is F" and " (X_2, X_4) is G."

Particularization/Projection (Transduction)

In test-score semantics, we usually deal with relations or possibility distributions which are both particularized and projected. For example:

$$X(w)R[X(s) \text{ is } F; X(v) \text{ is } G] \quad (3.18)$$

which should be interpreted as a fuzzy relation R which is first particularized and then projected. It should be noted, however, that if $s \subset w$ and $v \subset w$, then (3.18) could also be interpreted as a relation R which is first projected and then particularized. It is easy to show that the latter interpretation leads to the same result by virtue of the distributivity of \wedge (min) over \vee (max).

In what follows, we shall employ the suggestive term transduction to refer to the combination of particularization and projection. In essence, transduction may be viewed as a generalization of the familiar operation of finding the value of a function for a given value of its argument. In this light, (3.18) may be read as "substitute F for X(s), G for X(v) and get X(w)," with the understanding that the substitution of F for X(s) and G for X(v) involves in actuality the substitution of F and G for the possibility distributions of X(s) and X(v), and reading the possibility distribution of X(w). In particular, in the special case of relations of the form

$$X_{i_1} \times \dots \times X_{i_m} \times \mu R[X_{j_1} = a_1; \dots; X_{j_\ell} = a_\ell] \quad (3.19)$$

in which a_1, \dots, a_ℓ are specified values of the variables $X_{j_1}, \dots, X_{j_\ell}$, what is read is a fuzzy subset of $U_{i_1} \times \dots \times U_{i_m}$ which is a "section" of

R with the planes $X_{j_1} = a_1, \dots, X_{j_\ell} = a_\ell$. More particularly, an expression of the form

$$\mu R[X_1 = a_1; \dots; X_n = a_n] \quad (3.20)$$

should be interpreted as "Read (or get or obtain) the grade of membership of the n-tuple (a_1, \dots, a_n) in the fuzzy relation R."

As an illustration, the following expressions should be interpreted as indicated:

(a) $\text{Age}^{\text{POPULATION}}[\text{Name} = \text{Barbara}]$.

Obtain Barbara's age from the relation POPULATION which includes Name and Age among its variables. In this case, Barbara is transduced into her age.

(b) $\text{Name}_{2 \times \mu}^{\text{FRIEND}}[\text{Name1} = \text{Maria}]$.

Obtain the fuzzy set of Maria's friends from the fuzzy relation FRIEND in which μ is the degree to which Name2 is a friend of Name1. In this example, Maria is transduced into the fuzzy set of her friends.

(c) $\mu^{\text{FRIEND}}[\text{Name1} = \text{Lucia}; \text{Name2} = \text{Richard}]$.

Obtain from the relation FRIEND the degree to which Richard is a friend of Lucia. Here Lucia and Richard are transduced into their grade of friendship.

(d) $\text{Name}_{\times \mu}^{\text{POPULATION}}[\text{Age} = \text{YOUNG}]$.

Obtain from POPULATION the fuzzy set of names of those who are young. In this case, the fuzzy set YOUNG is transduced into a fuzzy subset of the nonfuzzy set $\text{Name}^{\text{POPULATION}}$. A point that should be noted is that even though the relation POPULATION is

nonfuzzy and has no μ attribute, the particularized relation POPULATION[$\Pi_{\text{Age}} = \text{YOUNG}$] is fuzzy and has a μ attribute which is the attribute referred to in Name $\times \mu$.

Cardinality of fuzzy sets

How many lakes are there in California? What is the proportion of tall men among fat men? What is the meaning of "Brian is much taller than most of Mildred's friends?" What is the denotation of several red apples? The answers to questions of this type hinge on the concept of cardinality of fuzzy sets or, more generally, on the concept of measure. In what follows, we shall give a definition of cardinality which serves to provide a basis for testing the elastic constraints induced by fuzzy quantifiers such as many, most, several, few, almost all, etc. The tests in question will be described in Sections 4 and 5.

As should be expected, the concept of cardinality of a fuzzy set is an extension of the count of elements of a crisp, i.e., nonfuzzy, set. Specifically, assume, for simplicity, that A is a fuzzy set expressed as

$$A = \mu_1/u_1 + \dots + \mu_n/u_n, \quad (3.21)$$

where the u_i , $i = 1, \dots, n$, are elements of a universe of discourse U. A simple way of extending the concept of cardinality which was suggested by DeLuca and Termini [24] and which is related to the notion of the probability measure of a fuzzy event [132] is to form the arithmetic sum of the grades of membership. We shall refer to this sum, with or

without a round-off to the nearest integer, as the sigma-count or, equivalently, as nonfuzzy cardinality of A. Thus, by definition,

$$\Sigma\text{Count}(A) \triangleq \sum_{i=1}^n \mu_i . \quad (3.22)$$

For example

$$\Sigma\text{Count} (0.6/a + 0.9/b + 1/c + 0.6/d + 0.2/e) = 3 .$$

A less simple but perhaps more natural extension which was suggested in [137] expresses the cardinality of a fuzzy set as a fuzzy number. Thus, let A_α be the α -level-set of A, i.e., the nonfuzzy set defined by

$$A_\alpha \triangleq \{u_i | \mu_A(u_i) \geq \alpha\}, \quad 0 < \alpha \leq 1, \quad u_i \in U, \quad i = 1, \dots, n , \quad (3.23)$$

where $\mu_i \triangleq \mu_A(u_i)$, $i = 1, \dots, n$, is the grade of membership of u_i in A. Then, as shown in [133], A may be expressed in terms of the A_α by the identity

$$A = \sum_{\alpha} \alpha A_\alpha , \quad (3.24)$$

where \sum stands for the union and αA_α is a fuzzy set whose membership function is defined by

$$\begin{aligned} \mu_{\alpha A_\alpha}(u) &= \alpha \quad \text{for } u \in A_\alpha \\ &= 0 \quad \text{elsewhere} . \end{aligned} \quad (3.25)$$

For example, if $U = \{a, b, c, d, e, f\}$ and

$$A = 0.6/a + 0.9/b + 1/c + 0.6/d + 0.2/e \quad (3.26)$$

then

$$A_1 = \{c\}$$

$$A_{0.9} = \{b,c\}$$

$$A_{0.6} = \{a,b,c,d\}$$

$$A_{0.2} = \{a,b,c,d,e\}$$

and (3.24) becomes

$$A = 1/c + 0.9/(b+c) + 0.6/(a+b+c+d) \\ + 0.2/(a+b+c+d+e).$$

Now, let $\text{Count}(A_\alpha)$ denote the count of elements of the nonfuzzy set A_α . Then, the FGCount of A , where F stands for fuzzy and G stands for greater than, is defined as the fuzzy number

$$\text{FGCount}(A) \triangleq \sum \alpha / \text{Count}(A_\alpha) \quad (3.27)$$

with the understanding that any gap in the $\text{Count}(A_\alpha)$ may be filled by a lower count with the same α . For example, for A defined by (3.26), we have

$$\text{FGCount}(A) = 1/1 + 0.9/2 + 0.6/4 + 0.2/5 \quad (3.28) \\ = 1/1 + 0.9/2 + 0.6/3 + 0.6/4 + 0.2/5 .$$

It is of some help in understanding the significance of (3.27) to interpret a term such as $0.6/4$ in (3.28) as the assertion:

The truth-value of the assertion that A contains at least 4 elements is 0.6.

More generally, let p_m , q_m and r_m be the propositions:

$p_m \triangleq$ A contains at least m elements

$q_m \triangleq$ A contains at most m elements

and

$r_m \triangleq$ A contains no more and no less than m elements.

Furthermore, assume that the elements of A are sorted in descending order, so that $\mu_m \leq \mu_k$ if $m \geq k$. Then, the truth-values of p_m , q_m and r_m are given by [140]

$$\text{Tr}\{p_m\} = \mu_m \tag{3.29}$$

$$\text{Tr}\{q_m\} = 1 - \mu_{m+1} \tag{3.30}$$

$$\text{Tr}\{r_m\} = \mu_m \wedge (1 - \mu_{m+1}) \tag{3.31}$$

These expressions provide a rationale for defining FGCount, FLCCount (L standing for less than), and FECount (E standing for equal to) as follows.

Let A^+ denote A sorted in descending order and let NA^+ denote the fuzzy number resulting from replacing the mth element in A^+ by μ_m/m and adding the element $1/0$. For example, if

$$A = 0.6/a + 0.9/b + 1/c + 0.6/d + 0.2/e \tag{3.32}$$

then

$$A^+ = 1/c + 0.9/b + 0.6/a + 0.6/d + 0.2/e \tag{3.33}$$

$$NA^+ = 1/0 + 1/1 + 0.9/2 + 0.6/3 + 0.6/4 + 0.2/5 . \tag{3.34}$$

In terms of this notation, the definition of $FGCount(A)$ stated earlier (3.27) may be expressed more succinctly as

$$FGCount(A) = NA+ . \quad (3.35)$$

In a similar vein, the definitions of $FLCount(A)$ and $FECCount(A)$ may be expressed as

$$FLCount(A) = (NA+)' \ominus 1 \quad (3.36)$$

and

$$FECCount(A) = FGCount(A) \cap FLCount(A) , \quad (3.37)$$

where $(NA+)'$ denotes the complement of $NA+$, \ominus represents fuzzy subtraction [136], [76], [26], and \cap is the operation of intersection.

A basic identity which relates the fuzzy cardinalities of A , B , $A \cap B$ and $A \cup B$ may be expressed as

$$FGCount(A \cup B) \oplus FGCount(A \cap B) = FGCount(A) \oplus FGCount(B) , \quad (3.38)$$

where \oplus denotes fuzzy addition. For example, if

$$A = 0.4/2 + 1/3 + 0.2/4$$

$$B = 0.5/3 + 1/4 + 0.3/5$$

then

$$A \cup B = 0.4/2 + 1/3 + 1/4 + 0.3/5$$

$$A \cap B = 0.5/3 + 0.2/4$$

$$FGCount(A) = 1/0 + 1/1 + 0.4/2 + 0.2/3$$

$$FGCount(B) = 1/0 + 1/1 + 0.5/2 + 0.3/3$$

$$FGCount(A) \oplus FGCount(B) = 1/0 + 1/1 + 1/2 + 0.5/3 + 0.4/4 + 0.3/5 + 0.2/6$$

$$(3.39)$$

$$\text{FGCount}(A \cup B) = 1/0 + 1/1 + 1/2 + 0.4/3 + 0.3/4$$

$$\text{FGCount}(A \cap B) = 1/0 + 0.5/1 + 0.2/2$$

and

$$\begin{aligned} \text{FGCount}(A \cup B) \oplus \text{FGCount}(A \cap B) &= 1/0 + 1/1 + 1/2 + 0.5/3 \\ &\quad + 0.4/4 + 0.3/5 + 0.2/6 \end{aligned}$$

in agreement with (3.39)

In formulating tests for cardinality in Sections 4 and 5, we shall be employing for the most part the definitions of ΣCount and FGCount . Although the definitions of fuzzy cardinality expressed by (3.27) and (3.35) are not simple enough to be obvious on first exposure, the examples presented in Section 5 suggest that the concept of fuzzy cardinality is a natural extension of the corresponding concept for crisp sets.

4. Test-Score Semantics: Meaning Representation

In the preceding section, we have discussed some of the basic concepts which underlie the testing of fuzzy relations in a relational database. In this and the following section our attention will be focused on the principal issues relating to the representation of the meaning of a proposition by the testing of constraints which are induced by it.

Let p be a proposition whose meaning we wish to represent. The first question that arises is: What is the collection of relational databases which should be used as the object of testing? Once the answer to this question is arrived at, the next question is: What are the tests to be performed and how should their test scores be aggregated?

With respect to the first question, the position we shall take is that the choice of the test bed should be goal-oriented, that is, should depend on the state of knowledge of the actual or composite addressee of the meaning-representation process. In plain language, what this means is that in representing the meaning of p we should be influenced by our perception of the concepts and variables which are explicit or implicit in p and whose meaning is known to the addressee. Generally, these are tacitly assumed to be the concepts whose labels appear in p , together with the attributes with which they are associated. However, in test score semantics, this is a flexible rather than a rigid desideratum.

As an illustration, consider the proposition $p \triangleq$ Overeating causes obesity, and assume that the intended interpretation of p is

$$q \triangleq \text{Most of those who overeat are obese .} \quad (4.1)$$

Furthermore, assume that the addressee knows, in principle, the meaning

of the terms most, overeat and obese, so that the objective of the meaning-representation process is a precisiation of the meaning of p. In this event, an appropriate set of relational frames for the database might be:

$$\begin{aligned} \text{DF1} \triangleq & \text{POPULATION}[\text{Name}; \text{Age}; \text{Weight}; \text{Height}; \text{Consumption}] \quad (4.2) \\ & + \text{OBESE}[\text{Age}; \text{Height}, \text{Weight}; \mu] \\ & + \text{OVEREAT}[\text{Consumption}; \mu] \\ & + \text{MOST}[r; \mu], \end{aligned}$$

where DF stands for database frame and + denotes the union.

The first relation, labeled POPULATION, lists the names of individuals, together with the values of the attributes Age, Weight, Height and Consumption, with the latter expressed as the ratio of the level of food consumption to what would be considered to be the normal level of consumption for an individual of that age, weight and height.

The relation OBESE defines the grade of membership of an individual, μ , in the fuzzy set of obese individuals as a function of the attributes Age, Weight, and Height. The relation OVEREAT defines the grade of membership of an individual, μ , in the fuzzy set of those who overeat, as a function of Consumption. The last relation, MOST, defines the fuzzy quantifier most as a fuzzy subset of the unit interval, with r representing a numerical proportion.

Alternatively, and more simply, we could assume that DF consists of the following relations:

$$\begin{aligned} \text{DF2} \triangleq & \text{POPULATION}[\text{Name}] \quad (4.3) \\ & + \text{OBESE}[\text{Name}; \mu] + \text{OVEREAT}[\text{Name}; \mu] + \text{MOST}[r; \mu]. \end{aligned}$$

In this case, the fuzzy subsets OBESE and OVEREAT of POPULATION are defined directly rather than through the intermediary of the numerically-valued attributes Age, Weight and Height. As should be expected, the representation of the meaning of p as a test on the database represented by (4.3) would be less informative than a test on (4.2).

As was alluded to already, the test on a database, D, depends on the choice of relational frames, DF. As an illustration, for the database frame DF2 defined by (4.3), the compatibility test for p and D may be described as follows.

1. Count the number of individuals in POPULATION who overeat.
To this end, let Name_i denote the name of ith individual in POPULATION. Using the expression for the Σ Count as defined by (3.22), we have

$$\Sigma\text{Count}(\text{OVEREAT}) = \sum_i (\mu_{\text{OVEREAT}}[\text{Name} = \text{Name}_i]). \quad (4.4)$$

2. Count the number of individuals in POPULATION who are obese and overeat.

In this case, we have to compute the Σ Count of the intersection of the fuzzy sets OVEREAT and OBESE. The grade of membership of Name_i in the intersection is given by (see [139])

$$\mu_{\text{OVEREAT} \cap \text{OBESE}}(\text{Name}_i) = \mu_{\text{OVEREAT}}(\text{Name}_i) \wedge \mu_{\text{OBESE}}(\text{Name}_i) \quad (4.5)$$

where

$$\mu_{\text{OVEREAT}}(\text{Name}_i) = \mu_{\text{OVEREAT}}[\text{Name} = \text{Name}_i] \quad (4.6)$$

and

$$\mu_{\text{OBESE}}(\text{Name}_i) = \mu_{\text{OBESE}}[\text{Name} = \text{Name}_i] . \quad (4.7)$$

Consequently, the ΣCount of individuals who are obese and overeat is given by

$$\begin{aligned} \Sigma\text{Count}(\text{OVEREAT} \cap \text{OBESE}) &= \sum_i (\mu_{\text{OVEREAT}}[\text{Name} = \text{Name}_i]) \quad (4.8) \\ &\quad \wedge (\mu_{\text{OBESE}}[\text{Name} = \text{Name}_i]) . \end{aligned}$$

3. Compute the proportion of those who are obese among the fuzzy subset of those who overeat. Using (4.6) and (4.8), we find

$$\begin{aligned} \gamma &\triangleq \frac{\Sigma\text{Count}(\text{OVEREAT} \cap \text{OBESE})}{\Sigma\text{Count}(\text{OVEREAT})} \quad (4.9) \\ &= \frac{\sum_i (\mu_{\text{OVEREAT}}[\text{Name}=\text{Name}_i]) \wedge (\mu_{\text{OBESE}}[\text{Name}=\text{Name}_i])}{\sum_i (\mu_{\text{OVEREAT}}[\text{Name}=\text{Name}_i])} . \end{aligned}$$

4. Compute the test score corresponding to the degree to which the proportion γ expressed by (4.9) satisfies the constraint induced by the fuzzy quantifier most. Using (4.9), the test score for this constraint is found to be given by

$$\tau = \mu_{\text{MOST}}[r=\gamma] . \quad (4.10)$$

This test score, then, may be interpreted as the truth of p given D or, equivalently, as the possibility of D given p .

There are several important observations relating to this example that are of general validity.

- (a) The meaning of p is represented by the test which yields the test score τ .
- (b) The description of the test involves only the relational frames in the assumed database and not the data. In other words, the test represents the intension of p [139].
- (c) The structure of the test depends on the choice of relational frames. Thus, the description of the test would be different for DF1 (defined by (4.2)). Furthermore, for the same choice of relational frames, different tests would be required to accommodate different definitions of cardinality. This point is discussed in greater detail in Example 4, Section 5.
- (d) The choice of DF affects the explanatory effectiveness of meaning representation in test-score semantics. More specifically, lessening the degree of detail in DF has the effect of lowering the degree of explanatory effectiveness. For example, in the case of DF1 and DF2, DF2 is less detailed than DF1. Correspondingly, the test procedure associated with DF2 conveys less information about the meaning of p than that associated with DF1.

The simplest possible DF for the proposition $p \triangleq$ Overeating causes obesity is

$$DF = \text{CAUSE}[\text{Cause}; \text{Effect}] . \quad (4.11)$$

For this DF, the test reduces to the containment condition

$$(\text{Overeat}, \text{Obese}) \subset \text{CAUSE}$$

which signifies that the tuple $(\text{Overeat}, \text{Obese})$ belongs to the relation CAUSE. Equivalently, the test may be represented as

$$\text{CAUSE}[\text{Cause} = \text{Overeat}; \text{Effect} = \text{Obese}] \quad (4.12)$$

which is similar in form to the conventional semantic-network representation of the meaning of p.

It is of interest to observe that the DF represented by (4.11) is insufficiently detailed to allow a differentiation between the meanings of the propositions

$p \triangleq$ Overeating causes obesity

and

$p^1 \triangleq$ Obesity is caused by overeating

with the latter interpreted as

$$q^1 \triangleq \text{Most of those who are obese overeat.} \quad (4.13)$$

It can readily be verified that the test score corresponding to q^1 is given by

$$\tau = \mu \text{MOST}[r = \frac{\Sigma \text{Count}(\text{OBESE} \cap \text{OVEREAT})}{\Sigma \text{Count}(\text{OBESE})}] \quad (4.14)$$

which differs from (4.9) in the denominator of r.

As a further example consider the proposition

$p \triangleq$ Dana is very young and Tandy is not much older than Dana

In this case, we shall assume that the database frame is:¹¹

$$\begin{aligned} \text{DF} \triangleq & \text{POPULATION}[\text{Name}; \text{Age}] & (4.15) \\ & + \text{YOUNG}[\text{Age}; \mu] \\ & + \text{MUCH. OLDER}[\text{Age1}; \text{Age2}; \mu] . \end{aligned}$$

¹¹To place in evidence that a sequence of words is a label of a relation, we employ the convention of inserting periods between the words.

In the last relation, μ is the degree to which Age1 is much older than Age2.

The proposition under consideration induces two elastic constraints:

(a) a constraint on the age of Dana, and (b) a constraint on the age of Tandy relative to that of Dana. To test these constraints, we proceed as follows.

1. Find the ages of Dana and Tandy. Using (4.15), we have

$$\text{Age (Dana)} = \text{Age}^{\text{POPULATION[Name = Dana]}} \quad (4.16)$$

$$\text{Age (Tandy)} = \text{Age}^{\text{POPULATION[Name = Tandy]}}. \quad (4.17)$$

2. Test the constraint on the age of Dana. Denoting the test score for this constraint by τ_1 , we have

$$\tau_1 = (\mu \text{YOUNG[Age=Age (Dana)]})^2, \quad (4.18)$$

where Age (Dana) is given by (4.16) and the squaring accounts for the effect of the modifier very (see (4.31)).

3. Test the constraint on the age of Tandy relative to that of Dana. The test score for this constraint is given by

$$\tau_2 = 1 - \mu \text{MUCH.OLDER[Age1 = Age (Tandy);} \quad (4.19)$$

$$\text{Age2 = Age (Dana),}$$

where the subtraction of the second term from unity accounts for the effect of the negation not in the relation "not much older." (See (4.30).)

4. Aggregate the test scores τ_1 and τ_2 . Using the product for aggregate (instead of the usual min), we arrive at the overall test score

$$\tau = \tau_1 \tau_2 \quad (4.20)$$

as a measure of the compatibility of p with the database.

Focused Translation

In general, the test score τ for a given test T depends not on the entire database D but on a subset of it. Typically, if X_1, \dots, X_m are the variables involved in D (i.e., the designations of entries in relations in D), then τ may depend on a proper subset of the X_i , say X_{i_1}, \dots, X_{i_k} . To take a simple example, if the database, D , consists of two relations, say POPULATION[Name; Age; Weight; Height] and YOUNG[Age; μ], then the compatibility of the proposition

$p \triangleq$ Lillian is young

with D depends only on Lillian's age -- which is an entry under Age in POPULATION -- and the degree to which Lillian's age satisfies the constraint induced by young -- which is an entry under μ in the relation YOUNG.

More generally, a subset $F(D,p)$ of D will be said to be a focus of D relative to p if

(a) The compatibility of p with D is identical with the compatibility of p with $F(D,p)$

and

(b) $F(D,p)$ is minimal, i.e., there is no proper subset of $F(D,p)$ with property (a).

The notion of a focus provides a natural point of departure for introducing the concept of a focused translation. Specifically, given p and D , let $F(D,p)$ be the focus of D relative to p and let X_{i_1}, \dots, X_{i_k} or, more simply, X_1, \dots, X_n , be the variables which enter

into $F(D,p)$. Then, using unfocused translation, we can compute for each D -- and hence for each n -tuple (u_1, \dots, u_n) of values of X_1, \dots, X_n -- the compatibility, τ , of p with $\{X_1 = u_1, \dots, X_n = u_n\}$. Now, if we interpret τ as the possibility of the n -tuple (u_1, \dots, u_n) , i.e.,

$$\tau = \text{Poss}\{X_1 = u_1, \dots, X_n = u_n\} \quad , \quad (4.21)$$

then the focused translation of p may be expressed symbolically as

$$p \rightarrow \Pi_{(X_1, \dots, X_n)} = F \quad , \quad (4.22)$$

where τ defines the membership function of F ; $\Pi_{(X_1, \dots, X_n)}$ represents the possibility distribution of the n -ary variable (X_1, \dots, X_n) ; the variables X_1, \dots, X_n are the base variables in p ; and the right-hand member of (4.22) is what we have referred to in Section 2 as the possibility assignment equation.

As a simple illustration of the concept of a focused translation, consider the proposition

$$p \triangleq \text{Brian is much taller than Mildred} \quad . \quad (4.23)$$

Assuming that DF consists of the relational frames

$$DF = \text{POPULATION}[\text{Name}; \text{Height}] + \\ \text{MUCH.TALLER}[\text{Height1}; \text{Height2}; \mu] \quad ,$$

the unfocused translation of p is characterized by the following test:

1. Determine the height of Brian and Mildred:

$$\text{Height (Brian)} = \text{Height}^{\text{POPULATION}}[\text{Name} = \text{Brian}]$$

$$\text{Height (Mildred)} = \text{Height}^{\text{POPULATION}}[\text{Name} = \text{Mildred}] .$$

2. Test the constraint induced by the fuzzy relation MUCH.TALLER.

The test score for this constraint is given by

$$\tau = \mu_{\text{MUCH.TALLER}}[\text{Height1} = \text{Height (Brian)} ; \quad (4.24) \\ \text{Height2} = \text{Height (Mildred)}] .$$

Since p induces just one constraint, no aggregation is necessary and τ as expressed by (4.24) defines the compatibility of p with D .

Correspondingly, the focused translation of p may be expressed compactly as

$$p \rightarrow \Pi(\text{Height(Brian)}, \text{Height(Mildred)}) = \text{MUCH.TALLER} . \quad (4.25)$$

In this form, the translation of p signifies that the base variables in p are $X_1 = \text{Height(Brian)}$ and $X_2 = \text{Height(Mildred)}$, and that the focused translation of p defines the meaning of p as an assignment statement which assigns the fuzzy relation MUCH.TALLER to the joint possibility distribution of X_1 and X_2 .

Additional examples of both unfocused and focused translations will be presented in Section 5. What is important to note at this juncture is that, as its name implies, a focused translation serves the purpose of placing in evidence the base variables in p and focuses the translation process on the determination of the joint possibility distribution of these variables. In general, a focused translation has the advantage of greater transparency in the case of relatively

simple propositions, but becomes rather unwieldy in the case of propositions whose DF's involve more than a few relational frames.

When focusing is employed in the translation of a complex proposition, it is frequently advantageous to (a) decompose it into simpler propositions; (b) translate separately the constituent propositions; and (c) compose the results. In this connection, it is convenient to have a collection of translation rules which may be employed -- when this is possible -- to compose the meaning of a proposition from the meanings of its constituents. Among the basic translation rules in PRUF which serve this purpose are the rules of Type I, Type II, Type III and Type IV.¹² For convenient reference, these rules are summarized in the following.

Translation Rules

Modifier rule (Type I). Let X be a variable which takes values in a universe of discourse U , and let F be a fuzzy subset of U . Consider the proposition

$$p \underline{\Delta} X \text{ is } F \quad (4.26)$$

or, more generally,

$$p \underline{\Delta} N \text{ is } F, \quad (4.27)$$

where N is a variable, an object or a proposition.

Now, if, in a particular context, the proposition $X \text{ is } F$ translates into

¹²A more detailed discussion of these rules may be found in [139].

$$X \text{ is } F \rightarrow \Pi_X = F \quad (4.28)$$

then in the same context

$$X \text{ is } mF \rightarrow \Pi_X = F^+, \quad (4.29)$$

where m is a modifier such as not, very, more or less, etc., and F^+ is a modification of F induced by m . More specifically: If $m = \text{not}$, then $F^+ \triangleq F' \triangleq$ complement of F , i.e.,

$$\mu_{F^+}(u) = 1 - \mu_F(u), \quad u \in U. \quad (4.30)$$

If $m = \text{very}$, then $F^+ = F^2$, i.e.,

$$\mu_{F^+}(u) = \mu_F^2(u), \quad u \in U. \quad (4.31)$$

If $m = \text{more or less}$, then $F^+ = \sqrt{F}$, i.e.,

$$\mu_{F^+}(u) = \sqrt{\mu_F(u)}, \quad u \in U. \quad (4.32)$$

As a simple illustration of (4.31), if SMALL is defined as

$$\text{SMALL} = 1/0 + 1/1 + 0.8/2 + 0.6/3 + 0.4/4 + 0.2/5$$

then

$$x \text{ is very small} \rightarrow \Pi_X = F^2, \quad (4.33)$$

where

$$F^2 = 1/0 + 1/1 + 0.64/2 + 0.36/3 + 0.16/4 + 0.04/5.$$

It should be noted that (4.30), (4.31) and (4.32) should be viewed as default rules which may be replaced by other translation rules in cases in which some alternative interpretations of the modifiers not, very and more or less are more appropriate.

Conjunctive, Disjunctive and Implicational Rules (Type II). If

$$X \text{ is } F \rightarrow \Pi_X = F \text{ and } Y \text{ is } G \rightarrow \Pi_Y = G, \quad (4.34)$$

where F and G are fuzzy subsets of U and V , respectively, then

$$(a) \ X \text{ is } F \text{ and } Y \text{ is } G \rightarrow \Pi_{(X,Y)} = F \times G, \quad (4.35)$$

where

$$\mu_{F \times G}(u,v) \triangleq \mu_F(u) \wedge \mu_G(v). \quad (4.36)$$

$$(b) \ X \text{ is } F \text{ or } Y \text{ is } G \rightarrow \Pi_{(X,Y)} = \bar{F} \cup \bar{G}, \quad (4.37)$$

where

$$\bar{F} \triangleq F \times V, \quad \bar{G} \triangleq U \times G \quad (4.38)$$

and

$$\mu_{\bar{F} \cup \bar{G}}(u,v) = \mu_F(u) \vee \mu_G(v). \quad (4.39)$$

$$(c) \ \text{If } X \text{ is } F \text{ then } Y \text{ is } G \rightarrow \Pi_{(Y|X)} = \bar{F}' \oplus \bar{G}, \quad (4.40)$$

where $\Pi_{(Y|X)}$ denotes the conditional possibility distribution of Y given X , and the bounded sum \oplus is defined by

$$\mu_{\bar{F}' \oplus \bar{G}}(u,v) = 1 \wedge (1 - \mu_F(u) + \mu_G(v)). \quad (4.41)$$

Note. In stating the implicational rule in the form (4.40), we have merely chosen one of the several alternative ways in which the conditional possibility distribution $\Pi_{(Y|X)}$ may be defined, each of which has some advantages and disadvantages depending on the application. A detailed discussion of this issue can be found in [5], [72] and [105].

As simple illustrations of (4.35), (4.37) and (4.40), if

$$F \triangleq \text{SMALL} = 1/1 + 0.6/2 + 0.1/3$$

$$G \triangleq \text{LARGE} = 0.1/1 + 0.6/2 + 1/3$$

then

$$\begin{aligned} X \text{ is small and } Y \text{ is large} &\rightarrow \Pi_{(X,Y)} \\ &= 0.1/(1,1) + 0.6/(1,2) + 1/(1,3) + 0.1/(2,1) \\ &\quad + 0.6/(2,2) + 0.6/(2,3) + 0.1/(3,1) \\ &\quad + 0.1/(3,2) + 0.1/(3,3) . \end{aligned}$$

$$\begin{aligned} X \text{ is small or } Y \text{ is large} &\rightarrow \Pi_{(X,Y)} \\ &= 1/(1,1) + 1/(1,2) + 1/(1,3) + 0.6/(2,1) + 0.6/(2,2) \\ &\quad + 1/(2,3) + 0.1/(3,1) + 0.6/(3,2) + 1/(3,3) \end{aligned}$$

and

$$\begin{aligned} \text{If } X \text{ is small then } Y \text{ is large} &\rightarrow \Pi_{(Y|X)} \\ &= 0.1/(1,1) + 0.6/(1,2) + 1/(1,3) + 0.5/(2,1) \\ &\quad + 1/(2,2) + 1/(2,3) + 1/(3,1) + 1/(3,2) + 1/(3,3) . \end{aligned}$$

Quantification Rule (Type III). If $U = \{u_1, \dots, u_N\}$, Q is a fuzzy quantifier such as many, few, several, all, some, most, etc., and

$$X \text{ is } F \rightarrow \Pi_X = F \tag{4.42}$$

then the proposition "QX are F" (e.g., "many X's are large") translates into

$$QX \text{ are } F \rightarrow \Pi_{\Sigma \text{Count}(F)} = Q \tag{4.43}$$

if the concept of nonfuzzy cardinality is employed. (See (3.22).)

As a simple example, if the quantifier several is defined as

$$\text{SEVERAL} = 0.4/2 + 0.6/3 + 1/4 + 1/5 + 1/6 + 0.7/7 + 0.2/8 \tag{4.44}$$

then

$$\text{Several } X\text{'s are large} \rightarrow \Pi_{\sum_i \mu_{\text{LARGE}}(u_i)} = \text{SEVERAL} . \tag{4.45}$$

Examples in the which the concept of fuzzy cardinality is employed will be considered in Section 5.

Truth Qualification Rule (Type IV). Let τ be a linguistic truth-value, e.g., very true, quite true, more or less true, etc. Such a truth-value may be regarded as a fuzzy subset of the unit interval which is characterized by a membership function $\mu_\tau : [0,1] \rightarrow [0,1]$.

A truth-qualified proposition, e.g., "It is τ that X is F," is expressed as "X is F is τ ." As shown in [10], the translation rule for such propositions is given by

$$X \text{ is } F \text{ is } \tau \rightarrow \Pi_X = F^+, \quad (4.46)$$

where

$$\mu_{F^+}(u) = \mu_\tau(\mu_F(u)) . \quad (4.47)$$

As an illustration, consider the truth-qualified proposition

Teresa is young is very true

which by (4.46), (4.47) and (4.31) translates into

$$\Pi_{\text{Age(Teresa)}} = \mu_{\text{TRUE}}^2(\mu_{\text{YOUNG}}) . \quad (4.48)$$

Now, if we assume that

$$\mu_{\text{YOUNG}}(u) = (1 + (\frac{u}{25})^2)^{-1}, \quad u \in [0,100] \quad (4.49)$$

and

$$\mu_{\text{TRUE}}(v) = v^2, \quad v \in [0,1]$$

then (4.47) yields

$$\mu_{\text{Age(Teresa)}} = (1 + (\frac{u}{25})^2)^{-4}$$

as the possibility distribution of the age of Teresa.

Probability Qualification Rule (Type IV). This rule applies to propositions of the general form "X is F is λ ," where X is a real-valued variable, F is a linguistic value of X, and λ is a linguistic value of likelihood (or probability), e.g., "X is small is not very likely." Unless stated to the contrary, λ is assumed to be a fuzzy subset of the unit interval [0,1] which is characterized by its membership function μ_λ , and the probability distribution of X is characterized by its probability density function p, i.e.,

$$\text{Prob}\{X \in [u, u+du]\} = p(u)du . \quad (4.50)$$

As shown in [139], the translation rule for probability-qualified propositions is expressed by

$$X \text{ is } F \text{ is } \lambda \rightarrow \pi(p) = \mu_\lambda \left[\int_U \mu_F(u) p(u) du \right] , \quad (4.51)$$

where $\pi(p)$ denotes the possibility that the probability density function of X is p, and the integral in the right-hand member of (4.51) represents the probability of the fuzzy event [132] "X is F." Thus, in the case of probability-qualified propositions, the proposition "X is F is λ " induces a possibility distribution of the probability density function of X.

As a simple illustration, consider the proposition

$$q \triangleq \text{Vickie is young is very likely.} \quad (4.52)$$

In this case, $X \triangleq \text{Age(Vickie)}$ and the right-hand member of (4.51) becomes

$$\pi(p) = \mu_{\text{LIKELY}}^2 \left[\int_0^{100} \mu_{\text{YOUNG}}(u) p(u) du \right] . \quad (4.53)$$

The translation rules stated above may be used in combination. For example, consider the proposition

$p \triangleq$ If X is not very large and Y is more or less small then
 Z is very very large.

In this case, by the application of (4.30), (4.31), (4.32), (4.35) and (4.40), we find that p induces a conditional possibility distribution of Z given X and Y , i.e., $\Pi_{(Z|X,Y)}$. The possibility distribution function of this distribution is given by

$$\pi_{(Z|X,Y)}(w|u,v) = 1 \wedge (1 - (1 - \mu_{\text{LARGE}}^2(u)) \wedge \mu_{\text{SMALL}}^{0.5}(v) + \mu_{\text{LARGE}}^4(w)) \quad (4.54)$$

where μ_{LARGE} and μ_{SMALL} denote, respectively, the membership functions of the denotations of large and small in p .

Vector test scores and presuppositions

Since an aggregated test score is basically a summary, it would be natural to expect that in some cases the degree of summarization which is associated with a single overall test score might be excessive. In such cases, then, a vector test score might be required to convey the meaning of a proposition correctly.

Among the cases which fall into this category are propositions with false presuppositions, as in the classical example $p \triangleq$ The King of France is bald. In this case, an attempt to associate a single test score or truth-value with p leads to difficulties which have been discussed at length in the literature [68]. In our view, a natural way of dealing with these difficulties is provided by the concept of a vector test score -- a concept which furnishes a general framework

for the analysis of presuppositions and related issues.

Let p be a given proposition and let p^* be a presupposition which is associated with p . Usually, but not necessarily, p^* asserts the existence of an object which is characterized by p . In a departure from the conventional point of view, we shall assume that existence is a matter of degree and hence that p^* is a fuzzy presupposition, i.e., a proposition whose compatibility with a database may be a number other than 0 or 1.

As a simple illustration, consider the proposition

$$p \triangleq \text{By far the richest man in France is bald.} \quad (4.55)$$

In this case,

$$p^* \triangleq \text{There exists by far the richest man in France} \quad (4.56)$$

is a fuzzy presupposition by virtue of the fuzziness of the predicate by far the richest man.

To apply test-score semantics to this proposition, assume that the DF contains the following relational frames

$$\begin{aligned} \text{DF} \triangleq & \text{POPULATION}[\text{Name}; \text{Wealth}; \mu\text{Bald}] + \\ & \text{BY.FAR.RICHEST}[\text{Wealth}; \text{Wealth2}; \mu] . \end{aligned} \quad (4.57)$$

In the first relation in (4.57), Wealth is interpreted as the net worth of Name and μBald is the degree to which Name is bald. In the second relation, Wealth2 is the wealth of the second richest man, and μ is the degree to which Wealth and Wealth2 qualify the richest man in France (who is assumed to be unique) to be regarded as by far the richest man in France.

To compute the compatibility of p with the database, we perform the following test.

1. Sort POPULATION in descending order of Wealth. Denote the result by POPULATION \downarrow and let Name $_i$ be the i th name in POPULATION \downarrow .
2. Determine the degree to which the richest man in France is bald:

$$\tau_1 \triangleq \mu_{\text{Bald}}^{\text{POPULATION} \downarrow} [\text{Name} = \text{Name}_1] . \quad (4.58)$$

3. Determine the wealth of the richest and second richest men in France:

$$w_1 \triangleq \text{Wealth}^{\text{POPULATION} \downarrow} [\text{Name} = \text{Name}_1]$$

$$w_2 \triangleq \text{Wealth}^{\text{POPULATION} \downarrow} [\text{Name} = \text{Name}_2] .$$

4. Determine the degree to which the richest man in France is by far the richest man in France:

$$\tau_2 \triangleq \mu_{\text{BY.FAR.RICHEST}} [\text{Wealth} = w_1; \text{Wealth}_2 = w_2] . \quad (4.59)$$

5. The overall test score is taken to be the ordered pair

$$\tau = (\tau_1, \tau_2) . \quad (4.60)$$

Thus, instead of aggregating τ_1 and τ_2 into a single test score, we maintain their separate identities in the overall test score. We do this because the aggregated test score

$$\tau = \tau_1 \wedge \tau_2$$

would be creating a misleading impression when τ_1 is small, that is, when the test score for the constraint on the existence of "by far the richest man in France" is low.

In the simple case which we have used as an example, the overall test score as expressed by (4.60) has only two components. In general, however, a proposition p may have a multiplicity of fuzzy presuppositions each of which may have to be represented by a component test score in the overall test score for p . For example, the proposition

$p \triangleq$ By far the richest man in France is much taller than most of his close friends

has at least two fuzzy presuppositions

$p_1^* \triangleq$ There exists by far the richest man in France

$p_2^* \triangleq$ By far the richest man in France has close friends

and hence the overall test score for p will have to have at least three components.

It is important to observe that the fuzzy presuppositions $p_1^*, p_2^*, \dots, p_m^*$ which are associated with a proposition p depend in an essential way on the formulation of the test of compatibility of p with the database. For example, consider the proposition

$p \triangleq$ By far the richest man in France is by far the tallest man in Paris.

In this case, depending on the way in which the test procedure is formulated, either one of the following propositions could be regarded as a fuzzy presupposition of p :

p_1^* = There exists by far the richest man in France

p_2^* = There exists by far the tallest man in Paris .

The issue of vector test scores has many ramifications which extend beyond the scope of the present paper. In what follows, we shall confine ourselves to a discussion of examples in which the fuzzy presuppositions are tacitly assumed to have perfect test scores and hence need not be considered in the computation of compatibility.

5. Examples of Translation

The examples considered in this section are intended to clarify some of the aspects of test-score semantics which were discussed in general terms in Sections 3 and 4. The examples are relatively simple and, for the most part, involve propositions. When appropriate, both focused and unfocused translations are presented.

1. Margaret is slim and very attractive

Assume that

$$\begin{aligned} DF \triangleq & \text{POPULATION}[\text{Name}; \text{Weight}; \text{Height}] + & (5.1) \\ & \text{SLIM}[\text{Weight}; \text{Height}; \mu] + \\ & \text{ATTRACTIVE}[\text{Name}; \mu] . \end{aligned}$$

The steps in the test procedure are:

1. Find Margaret's height and weight:

$$a \triangleq \text{Weight}(\text{Margaret}) = \text{Weight}^{\text{POPULATION}[\text{Name} = \text{Margaret}]}$$

$$b \triangleq \text{Height}(\text{Margaret}) = \text{Height}^{\text{POPULATION}[\text{Name} = \text{Margaret}]} .$$

2. Test the constraint induced by SLIM:

$$\tau_1 \triangleq \mu \text{SLIM}[\text{Weight} = a; \text{Height} = b] . \quad (5.2)$$

3. Test the constraint induced by ATTRACTIVE:

$$\tau_2 \triangleq \mu \text{ATTRACTIVE}[\text{Name} = \text{Margaret}] . \quad (5.3)$$

4. Modify τ_2 to account for the modifier very:

$$\tau_3 \triangleq \tau_2^2 . \quad (5.4)$$

5. Aggregate τ_1 and τ_3 :

$$\tau = \tau_1 \wedge \tau_3 . \quad (5.5)$$

The aggregated test score given by (5.5) represents the compatibility of the proposition in question with the database whose DF is expressed by (5.1).

2. Ellen resides in a small city near Oslo.

Unfocused translation. Assume that

$$\begin{aligned} DF \triangleq & \text{RESIDENCE}[\text{Name}; \text{City.Name}] + & (5.6) \\ & \text{POPULATION}[\text{City.Name}; \text{Population}] + \\ & \text{SMALL}[\text{Population}; \mu] + \\ & \text{NEAR}[\text{City.Name1}; \text{City.Name2}; \mu] . \end{aligned}$$

1. Find the name of the residence of Ellen:
 - a $\triangleq \text{City.Name}^{\text{RESIDENCE}[\text{Name} = \text{Ellen}]}$.
2. Find the population of the residence of Ellen:
 - b $\triangleq \text{Population}^{\text{POPULATION}[\text{City.Name} = a]}$.
3. Test the constraint induced by SMALL:

$$\tau_1 \triangleq \mu^{\text{SMALL}[\text{Population} = b]} . \quad (5.7)$$

4. Test the constraint induced by NEAR:

$$\tau_2 = \mu^{\text{NEAR}[\text{City.Name1} = \text{Oslo}, \text{City.Name2} = a]} . \quad (5.8)$$

5. Aggregate τ_1 and τ_2 :

$$\tau = \tau_1 \wedge \tau_2 . \quad (5.9)$$

Focused translation. Suppose that we are interested in the location of residence of Ellen and that the relation RESIDENCE does not contain Ellen's name. Then, if the base variable implicit in the proposition under consideration is taken to be $X \triangleq \text{Location}(\text{Residence}(\text{Ellen}))$, the proposition translates into the possibility assignment equation

$$\begin{aligned} \Pi_{\text{Location}(\text{Residence}(\text{Ellen}))} = & \left(\text{City.Name2} \times_{\mu} \text{NEAR}[\text{City.Name1} = \text{Oslo}] \right) \cap \quad (5.10) \\ & \left(\text{City.Name} \times_{\mu} \text{POPULATION}[\Pi_{\text{Population}} = \text{SMALL}] \right) . \end{aligned}$$

In effect, (5.10) conveys the information that the possibility distribution of X is the intersection of two possibility distributions: the first reflects the constraint that the residence of Ellen is near Oslo while the second reflects the constraint that it is a small city.

3. Gary earns much more than his youngest brother.

Assume that

$$\begin{aligned} \text{DF } \triangleq & \text{POPULATION}[\text{Name}; \text{Income}; \text{Age}] + \quad (5.11) \\ & \text{BROTHER}[\text{Name1}; \text{Name2}] + \\ & \text{MUCH.MORE}[\text{Income1}; \text{Income2}; \mu] . \end{aligned}$$

1. Find Gary's income:

$$a \triangleq \text{Income} \text{POPULATION}[\text{Name} = \text{Gary}] .$$

2. Determine the set of Gary's brothers;

$$b \triangleq \text{Name1} \text{BROTHER}[\text{Name2} = \text{Gary}] .$$

3. Restrict POPULATION to brothers of Gary:

$$c \triangleq \text{POPULATION}[\text{Name} = \text{con}(b)] ,$$

where the prefix *con* indicates that *b* should be interpreted as a conjunctive fuzzy set (see (3.11)).

4. Find the income of Gary's youngest brother:

$$d \triangleq \text{Income}^{\text{Min Age}}(c), \quad (5.12)$$

where the operation $\text{Income}^{\text{Min Age}}$ finds the tuple in *c* which minimizes the value of *Age* and reads the *Income* value in this tuple.

5. Test the constraint induced by MUCH.MORE:

$$\tau = \mu_{\text{MUCH.MORE}}[\text{Income1} = a; \text{Income2} = d]. \quad (5.13)$$

This value of τ is the desired compatibility of the proposition with the database.

4. Several large balls.

In this case, the problem is to determine the compatibility of the description $d \triangleq$ several large balls with an object, *D*, which consists of a collection of *n* balls of various sizes represented by the DF

$$\begin{aligned} \text{DF} \triangleq & \text{BALL}[\text{Identifier}; \text{Size}] + & (5.14) \\ & \text{LARGE}[\text{Size}; \mu] + \\ & \text{SEVERAL}[N; \mu]. \end{aligned}$$

In (5.14), the first relation has *n* rows and is a listing of the identifiers of the balls and their respective sizes. In *SEVERAL*, μ is the degree to which an integer *N* fits the definition of several.

The description $d \triangleq$ several large balls is susceptible of different interpretations. In one, which we shall analyze first, the interpretation

is compartmentalized in the sense that the constraints induced by LARGE and SEVERAL are tested separately. In another interpretation, which will be referred to as integrated, the tests are not separated. To differentiate between these interpretations, we shall write [several] [Large] balls and [several large] balls to represent the first and second interpretations, respectively.

In an expanded form, the compartmentalized interpretation of d may be expressed as:

$$[\text{several}][\text{large}] \text{ balls} \Leftrightarrow \text{the object consists of several balls and all of the balls are large.} \quad (5.15)$$

The test procedure corresponding to this interpretation is the following.

1. Test the constraint induced by SEVERAL:

$$\tau_1 \triangleq \mu \text{SEVERAL}[N = n].$$

2. Find the size of the smallest ball:

$$a \triangleq \text{Size}^{\text{MinSize}}(\text{BALL}).$$

3. Test the constraint induced by LARGE by finding the degree to which the smallest ball is large:

$$\tau_2 \triangleq \mu \text{LARGE}[\text{Size} = a].$$

4. Aggregate the test scores:

$$\tau = \tau_1 \wedge \tau_2. \quad (5.16)$$

In the case of the integrated interpretation, the expanded form of d is assumed to be expressed as:

$$d \Leftrightarrow \text{at least several large balls and at most several large balls .} \quad (5.17)$$

Furthermore, we shall employ the FGCount and the FLCount to count the elements of D.

As a first step in the translation of d, we represent d as a conjunction of d_1 and d_2 , where

$$d_1 \triangleq \text{at least several large balls} \quad (5.18)$$

and

$$d_2 \triangleq \text{at most several large balls .} \quad (5.19)$$

Consider the particularized fuzzy set (see (3.11))

$$D_L = \text{BALL}[\text{Size} = \text{LARGE}] \quad (5.20)$$

which represents the restriction of the set BALL to large balls.

The FGCount of this set is obtained by sorting D_L in order of decreasing μ and replacing the i th element by i (see (3.35)). Thus,

$$\text{FGCount}(D_L) = ND_L + . \quad (5.21)$$

Now the quantifier at least several may be expressed as the composition of the binary relation \geq with SEVERAL. Thus, if

$$\text{SEVERAL} = 0.5/3 + 1/4 + 1/5 + 1/6 + 0.5/7$$

then

$$\geq \circ \text{SEVERAL} = 0.5/3 + 1/4 + 1/5 + \dots$$

and similarly, for at most several, we have

$$\leq \circ \text{SEVERAL} = 1/0 + \dots + 1/6 + 0.5/7 ,$$

where \circ denotes the composition operator (see [133]).

In terms of $\text{FGCount}(D_L)$, $\text{FLCount}(D_L)$ and the quantifiers $\geq \circ \text{SEVERAL}$ and $\leq \circ \text{SEVERAL}$, the test scores for the constraints induced by d_1 and d_2 may be expressed as¹³

$$\tau_1 \triangleq \sup(\text{FGCount}(D_L) \cap (\geq \circ \text{SEVERAL})) \quad (5.22)$$

and

$$\tau_2 \triangleq \sup(\text{FLCount}(D_L) \cap (\leq \circ \text{SEVERAL})) . \quad (5.23)$$

The aggregated test score, then, is given by

$$\tau = \tau_1 \wedge \tau_2 . \quad (5.24)$$

Note. It may be argued that (5.18) and (5.19) should not be treated as independent propositions, that is, as propositions in which the base variables are not jointly constrained. If we constrain the base variables to have the same value, the expression for the aggregated test score becomes

$$\tau = \sup(\text{FECCount}(D_L) \cap \text{SEVERAL}) \quad (5.25)$$

in which the $\text{FECCount}(D_L)$ (see (3.37)) may be normalized by scaling its membership function by the reciprocal of $\sup(\text{FECCount}(D_L))$.

¹³If F is a fuzzy set, $\sup(F)$ is its height, i.e., the supremum of $\mu_F(u)$ over U . (See [141].)

5. Let G be a given set of balls of various sizes. The proposition, p , which we wish to translate is related to the description considered in the preceding example. Specifically,

$p \triangleq G$ contains several large balls

In this case, DF is assumed to be:

$$\begin{aligned} DF \triangleq & G[\text{Identifier}; \text{Size}] + & (5.26) \\ & \text{LARGE}[\text{Size}; \mu] + \\ & \text{SEVERAL}[N; \mu]. \end{aligned}$$

1. Form the fuzzy subset of large balls in G :

$$a \triangleq G[\text{Size} = \text{LARGE}].$$

2. Determine the $FGCount$ of a :

$$b \triangleq FGCount(a).$$

3. The test score for the constraint induced by $SEVERAL$ and the relation of containment is given by (as in (5.22)).

$$\tau = \sup(b \cap (\geq \circ SEVERAL)). \quad (5.27)$$

6. Patricia has many acquaintances and a few close friends most of whom are highly intelligent.

The database frame is assumed to be:

$$\begin{aligned} DF \triangleq & \text{ACQUAINTANCE [Name1; Name2; } \mu] + & (5.28) \\ & \text{FRIEND [Name1; Name2; } \mu] + \\ & \text{INTELLIGENT [Name; } \mu] + \\ & \text{MANY [N; } \mu] + \\ & \text{FEW [N; } \mu] + \\ & \text{MOST [\rho; } \mu] . \end{aligned}$$

In ACQUAINTANCE, μ is the degree to which Name2 is an acquaintance of Name1, and likewise for FRIEND. In INTELLIGENT, μ is the degree to which Name is intelligent. Highly intelligent will be interpreted as INTELLIGENT³ and close friend as FRIEND², where the exponent represents the power to which μ is raised. For simplicity, we shall employ the sigma-count for the representation of the meaning of MANY, FEW and MOST.

1. Find the fuzzy set of Patricia's acquaintances:

$$a \triangleq \text{Name2}_{\times\mu} \text{ACQUAINTANCE [Name1 = Patricia]} .$$

2. Count the number of Patricia's acquaintances. Using the sigma-count, we have:

$$b \triangleq \Sigma\text{Count}(a) .$$

3. Find the test score for the constraint induced by MANY:

$$\tau_1 = \mu \text{MANY [N = b]} .$$

4. Find the fuzzy set of friends of Patricia:

$$c \triangleq \text{Name2} \times_{\mu} \text{FRIEND} [\text{Name1} = \text{Patricia}] .$$

5. Find the set of close friends by intensifying FRIEND:

$$d = c^2 .$$

6. Determine the count of d:

$$e \triangleq \Sigma \text{Count}(c^2) .$$

7. Find the test score for the constraint induced by FEW:

$$\tau_2 \triangleq \mu_{\text{FEW}} [N = e] .$$

8. Find the set of close friends of Patricia who are highly intelligent:

$$f = d \cap \text{INTELLIGENT}^3 .$$

9. Determine the count of f:

$$g = \Sigma \text{Count}(f) .$$

10. Form the proportion of those who are highly intelligent among the close friends of Patricia:

$$r = \frac{\Sigma \text{Count}(f)}{\Sigma \text{Count}(d)} = \frac{g}{e} .$$

11. Find the test score for the constraint induced by MOST:

$$\tau_3 = \mu_{\text{MOST}} [\rho = r] .$$

12. The aggregated test score for the proposition under consideration is given by

$$\tau = \tau_1 \wedge \tau_2 \wedge \tau_3. \quad (5.29)$$

7. During the past few months three large tankers carried a total of 500,000 tons of oil to Naples.

The database frame is assumed to be:

$$\begin{aligned} DF \triangleq & \text{TANKER [Name; Displacement; Cargo; Weight; Destination; Time.Arrival]} + \\ & \text{LARGE [Displacement; } \mu] + \\ & \text{FEW.MONTHS [t; } \mu] \\ & \text{500,000 [N; } \mu] . \end{aligned} \quad (5.30)$$

The inclusion of the relation 500,000 [N; μ] in the database reflects the assumption that the number 500,000 should be interpreted in an approximate rather than exact sense. Thus, the relation in question defines the degree to which a real number N fits the description "500,000." In the relation FEW.MONTHS, t stands for the time-difference between the present time and the time of arrival.

1. Particularize TANKER by specifying the displacement, cargo and destination. Thus (see (3.10)),

$$\begin{aligned} \text{TANKER1} \triangleq & \text{TANKER [Displacement = dis(LARGE);} \\ & \text{Cargo = Oil; Destination = Naples]} . \end{aligned}$$

2. To take into consideration the constraint induced by the number of tankers, we pick an arbitrary three-element subset of tankers, say

$$T3 = \{\text{Name}_i, \text{Name}_j, \text{Name}_k\}$$

and restrict TANKER1 to T3. Thus

$$\text{TANKER2} \triangleq \text{TANKER1 [Name = T3]} .$$

3. For each tanker in TANKER2 find the time of arrival and weight of cargo:

$$t_i \triangleq \text{Time.Arrival}^{\text{TANKER2}}[\text{Name} = \text{Name}_i]$$

$$c_i \triangleq \text{Weight}^{\text{TANKER2}}[\text{Name} = \text{Name}_i]$$

and likewise for Name_j and Name_k .

4. Determine the test score for the constraint induced by 500,000:

$$\tau_1 = \mu^{500,000}[N = c_i + c_j + c_k].$$

5. For each tanker in T3 determine the test score for the temporal constraint induced by FEW ($t_0 \triangleq$ present time):

$$\tau_{i2} = \mu^{\text{FEW}}[t = t_0 - t_i]$$

and likewise for t_j and t_k .

6. Aggregate the test scores determined in 5:

$$\tau_2 = \tau_{i2} \wedge \tau_{j2} \wedge \tau_{k2}$$

7. Aggregate τ_1 and τ_2 :

$$\tau_3 = \tau_1 \wedge \tau_2 \quad (5.31)$$

8. The test score expressed by (5.31) represents the compatibility of the given proposition with the subset TANKER2. To find the compatibility with the whole database, it is necessary to maximize τ_3 over all 3-element subsets of TANKER1, finding that subset which yields the best fit of the proposition to the database. Thus, the overall test score for the

proposition in question is given by

$$\tau = \text{Max}_{i,j,k} \tau_3 . \quad (5.32)$$

8. Our last example involves a command rather than a proposition. Specifically, what we wish to translate is:

$c \triangleq$ Keep under refrigeration,

in which the underlying assumption is that an item A (say a carton of milk), must be stored in a refrigerator when not in use. We assume that A is taken out of the refrigerator at times t_1, \dots, t_n , with $[t_i, t_i + d_i]$ representing the i th time-interval during which A is not under refrigeration. The ambient temperature during the time-interval $[t_i, t_i + d_i]$, $i = 1, \dots, n$, is assumed to be a_i .

In general, to translate a command, c , it is necessary to identify the compliance criterion, cc , which is implicit in c , and devise a procedure for testing the constraints induced by cc . To this end, assume that ed_i , $i = 1, \dots, n$, is the effective duration of noncompliance which takes into consideration the ambient temperature a_i . Thus

$$ed_i = g(d_i, a_i) ,$$

where g is a specified function.

The compliance criterion, cc , is assumed to be expressed by the proposition:

$cc \triangleq$ Total effective duration of nonrefrigeration is not much longer than K ,

where K is a specified length of time and

$$\text{Total effective duration } \underline{\Delta} \text{ ted } \underline{\Delta} \text{ ed}_1 + \dots + \text{ed}_n . \quad (5.33)$$

To translate cc, we assume that

$$\text{DF } \underline{\Delta} \text{ PROCESS[Effective.Duration] + MUCH.LONGER[T; } \mu \text{] ,} \quad (5.34)$$

in which the relation PROCESS lists the effective duration of noncompliance at times t_1, \dots, t_n , and MUCH.LONGER defines the degree to which T is much longer than K.

To compute the test score associated with cc we proceed as follows.

1. Obtain from the relation PROCESS the total effective duration:

$$\text{ted} = \sum_i (\text{Effective.Duration}_i)$$

2. Compute the test score:

$$\tau = 1 - \mu \text{ MUCH.LONGER[T = ted] .} \quad (5.35)$$

This test score, then, represents the degree to which an execution sequence defined by the relation PROCESS complies with the instruction $c \underline{\Delta}$ Keep under refrigeration.

6. Concluding Remark

To give an adequate idea of the applicability of test-score semantics to the problem of meaning representation in natural languages would require a far greater number of diverse examples than could be included in the present paper. In particular, with a few exceptions, we have not considered linguistic entities other than propositions and have not illustrated the use of truth-qualification, probability-qualification and possibility-qualification. Furthermore, we have not touched upon (a) the important issue of nesting of linguistic entities, and (b) the concepts of semantic equivalence and entailment. In sum, what we have attempted to convey is a general idea of the conceptual framework of test-score semantics and to articulate the conviction that a comprehensive theory of natural languages cannot be constructed without coming to grips with the issues of imprecision, elasticity and lack of specificity -- issues which are intimately related to the necessity for gradation of truth, membership and possibility.

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