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EFFICIENT CALCULATIONS ON POINTS AND LINES IN THE EUCLIDEAN PLANE

by

John H. Munson

Artificial Intelligence Group

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ABSTRACT

Calculations involving points and lines in Euclidean spaces are often subject to annoying anomalies when the trigonometric functions are invoked. This paper illustrates a method by which use of these functions may often be avoided, and presents explicit formulas for four basic calculations involving points and lines in two dimensions.

INTRODUCTION

In a variety of disciplines embracing the physical sciences and engineering, it is commonly necessary to perform calculations dealing with points and straight lines in two- or higher-dimensional Euclidean space. The points and lines may be nuclear events and tracks recorded on film, target points and paths of moving vehicles or equipment, crystal loci and lattice vectors, and so on. In each case, certain generic calculations are often required, such as the shortest (or perpendicular) distance between a point and a line, or the angle between two lines.

For example, how does one calculate the angle θ between two lines in the (x,y) Euclidean plane, assuming that the lines are specified by two points each: $\{A,B\}$ and $\{C,D\}$? The most common or obvious prescription is probably:

$$\theta_1 = \arctan((YB-YA)/(XB-XA))$$

$$\theta_2 = \arctan((YD-YC)/(XD-XC))$$

$$\theta = \theta_2 - \theta_1 \quad ,$$

where XA is the x coordinate of point A, and so on.

This prescription may be criticized on two grounds. First, it entails two calls to a relatively time-consuming trigonometric function evaluation. Secondly, and more important, the trigonometric functions bring with them certain well-known difficulties: infinite slopes (or tangents), ambiguities of sign, and ambiguities modulo 180° and 360°. The author has seen programs in which a calculation has been preceded by a tedious case analysis and a complete interchange of x and y values in order to combat such difficulties.

This paper illustrates a method for performing this and related calculations, which can be used in many cases to avoid the unnecessary introduction of explicit trigonometric functions. The method involves the direct use of vector products. The resulting programs are generally more compact and thus easier to code, debug, and interpret.

We exhibit the solutions to four basic problems in the two-dimensional (x,y) plane. The solutions can readily be translated by inspection into computer codes suitable for individual needs. Although analogous formulas can be derived in three (and higher) dimensions, they are more cumbersome and probably of less general interest.

THE DISTANCE FROM A POINT TO A LINE AND ALONG THE LINE

Let points A and B define a directed line segment in the (x,y) plane; i.e., a vector with its tail at A and its head at B (Figure 1). Given a third point, C, we desire the distance W of the perpendicular projection of C onto AB; we also find the distance V of the projection along the line AB. The formulas are:

$$\text{DOT} = (XC-XA)(XB-XA) + (YC-YA)(YB-YA)$$

$$\text{CROSS} = (YC-YA)(XB-XA) - (XC-XA)(YB-YA)$$

$$\text{DISTAB} = ((XB-XA)^2 + (YB-YA)^2)^{\frac{1}{2}}$$

$$V = \text{DOT/DISTAB}$$

$$W = \text{CROSS/DISTAB} \quad .$$

We shall not prove the correctness of the formulas. The reader versed in vector mathematics will recognize DOT and CROSS, respectively, as the dot (or inner) product and the (z component of the) cross product of the vectors AB and AC. The formulas are valid in all cases, assuming only that A and B are distinct points. V and W are both positive in the illustration (Figure 1). V will be negative if the projection onto AB (extended) falls behind A; W will be negative if C lies on the other side of AB.

Another interpretation of the formulas is that they transform point C to a new (v,w) coordinate system in which vector AB defines the primary axis. The formulas have been used, for example, to transform points on randomly oriented shallow curves (bubble-chamber tracks) to axes grossly aligned with the curves so that they may then be fitted by polynomial approximations.

In this and the examples that follow, the common element is the use of the dot and cross products of vectors with a given vector in the problem (always denoted AB). This method allows all distances and positions to be expressed in terms of components along, and perpendicular to, AB, thus avoiding explicit reference to angles in the (x,y) coordinate system.

THE REFLECTION OF A POINT ACROSS A LINE

In Figure 2, point R is the reflection of point C across the line AB (in the sense of a mirror image). Using the terminology of the preceding problem, R satisfies the conditions that its v coordinate is equal to that of C, and its w coordinate is the negative of C's. Setting up the equations for these conditions, and solving for the x and y coordinates of R, we obtain

$$XR = XA + ((XB-XA)\ DOT + (YB-YA)\ CROSS)/(DISTAB)^2$$

$$YR = YA + ((YB-YA)\ DOT - (XB-XA)\ CROSS)/(DISTAB)^2 \quad ,$$

where DOT, CROSS, and DISTAB are as defined previously.

THE ANGLE BETWEEN TWO LINES

In Figure 3, points A and B specify one directed line segment, points C and D, another. It is desired to find the angle θ measured counterclockwise from vector AB to vector CD or, in many cases, to find $\sin\theta$ and $\cos\theta$.

We define three intermediate quantities, analogous to the foregoing ones, which are (respectively) the dot product, cross product, and product of lengths of the two vectors:

$$DOT2 = (XD-XC)(XB-XA) + (YD-YC)(YB-YA)$$

$$CROSS2 = (YD-YC)(XB-XA) - (XD-XC)(YB-YA)$$

$$PROD = ((DOT2)^2 + (CROSS2)^2)^{\frac{1}{2}} \quad .$$

Then, from well-known vector relations,

$$\cos\theta = DOT2/PROD$$

$$\sin\theta = CROSS2/PROD \quad .$$

θ itself may be determined from these values. If $\sin\theta$ and/or $\cos\theta$ are the desired quantities, as is often the case, three or four trigonometric function evaluations are saved relative to the procedure outlined in the Introduction. The formulas work in all instances, assuming that A and B are distinct and that C and D are distinct.

THE INTERSECTION OF TWO LINES

It is desired to find the point E at which two vectors AB and CD, possibly extended, meet (Figure 4). We consider vectors AC, AD, and AE (not shown). Taking the dot and cross products of AC and AD with AB,

$$\text{DOTC} = (XC-XA)(XB-XA) + (YC-YA)(YB-YA)$$

$$\text{CROSSC} = (YC-YA)(XB-XA) - (XC-XA)(YB-YA)$$

$$\text{DOTD} = (XD-XA)(XB-XA) + (YD-YA)(YB-YA)$$

$$\text{CROSSD} = (YD-YA)(XB-XA) - (XD-XA)(YB-YA) \quad .$$

Now, since E is collinear with C and D, vector AE is a linear combination of AC and AD. Thus, the various vector products with AB are linearly related:

$$\frac{\text{DOTE-DOTC}}{\text{DOTD-DOTC}} = \frac{\text{CROSSE-CROSSC}}{\text{CROSSD-CROSSC}} \quad ,$$

where DOTE and CROSSE are, respectively, the dot and cross products of AE with AB. But CROSSE = 0, since point E lines on vector AB. Thus,

$$\text{DOTE} = \text{DOTC} - \text{CROSSC(DOTD-DOTC)/(CROSSD-CROSSC)} \quad .$$

Furthermore, we observe that

$$\text{ABSQUARE} = (XB-XA)^2 + (YB-YA)^2$$

is the dot product of AB with itself. Since AE is collinear with AB, AE's x and y components must bear the same proportion to those of AB as do the dot products, DOTE to ABSQUARE. Solving the resulting equations,

$$XE = XA + \text{DOTE}(XB-XA)/\text{ABSQUARE}$$

$$YE = YA + \text{DOTE}(YB-YA)/\text{ABSQUARE} \quad ,$$

which give the coordinates of E in terms of previously calculated quantities.

This solution works in all instances except those in which AB and CD are parallel and have no intersection, in which case (CROSSD-CROSSC) equals zero.

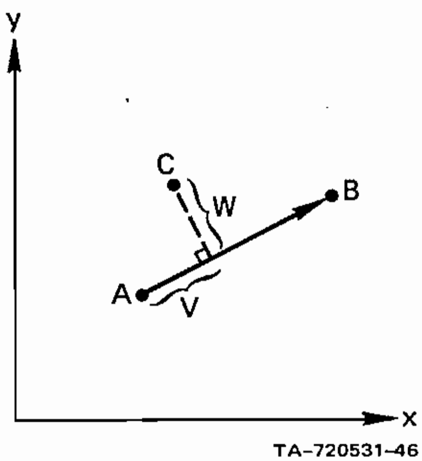

FIGURE 1 DISTANCE FROM A POINT TO A LINE AND ALONG THE LINE

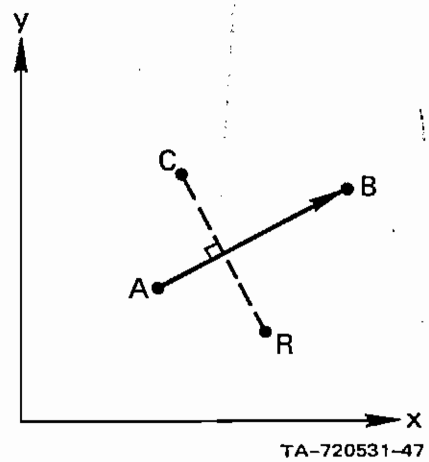

FIGURE 2 REFLECTION OF A POINT ACROSS A LINE

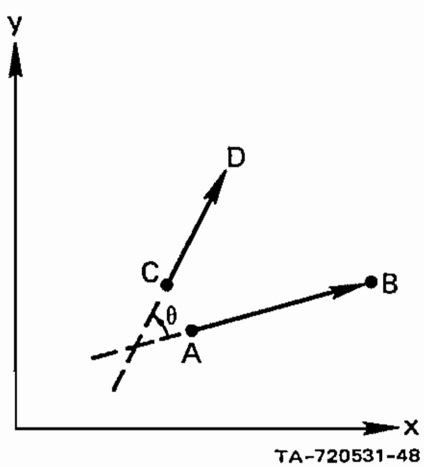

FIGURE 3 ANGLE BETWEEN TWO LINES

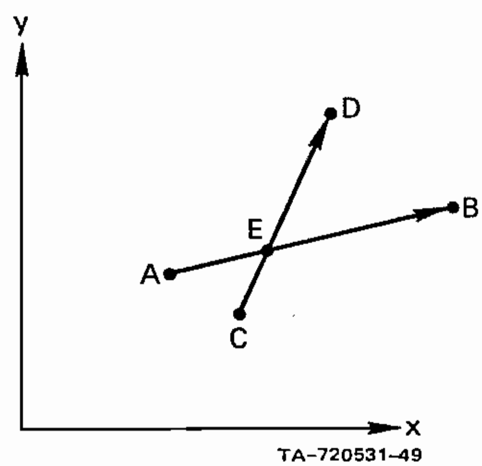

FIGURE 4 INTERSECTION OF TWO LINES