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FROM IMAGE IRRADIANCE TO SURFACE ORIENTATION

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By: Grahame B. Smith, Computer Scientist

Artificial Intelligence Center
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333 Ravenswood Ave. • Menlo Park, CA 94025
(415) 326-6200 • TWX: 910-373-2046 • Telex: 334-486

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Grahame B. Smith

Artificial Intelligence Center, SRI International
Menlo Park, California 94025

ABSTRACT

The image irradiance equation constrains the relationship between surface orientation in a scene and the irradiance of its image. This equation requires detailed knowledge of both the scene illumination and the reflectance of the surface material. For this equation to be used to recover surface orientation from image irradiance, additional constraints are necessary. The constraints usually employed require that the recovered surface be smooth. We demonstrate that smoothness is not sufficient for this task.

A new formulation of shape from shading is presented in which surface orientation is related to image irradiance without requiring detailed knowledge of the scene illumination, or of the albedo of the surface material. This formulation, which assumes isotropic scattering, provides some interesting performance parallels to those exhibited by the human visual system.

1 INTRODUCTION

Most previous work [1-8] on the problem of recovering surface shape from image shading has been based on solving the image irradiance equation, which relates the radiance of a scene to the irradiance of its image [1,2].[1] This formulation of the relationship between scene radiance and image irradiance is embodied in a first-order partial differential equation expressing scene depth as a function of image coordinates. Such a formulation requires specific knowledge of not only the reflectance characteristics of the surfaces in the scene, but also the position and strength of illumination sources. The approaches to solving this differential equation have generally been either by direct integration [1] or through an iterative algorithm that attempts to reduce the difference between the predicted image irradiance and the measured value [5-7]. Our interest is in the iterative approach because the alternative to it — direct integration — requires specific boundary conditions that are generally unknown (in natural scenes), and its behavior when applied to noisy pictures, is uncertain.

As the image irradiance equation is a single equation relating image irradiance and two independent variables (specifying surface orientation), it does not uniquely determine the two independent variables for a given value of image irradiance. Consequently, when this equation is used to recover surface shape additional constraints are necessary. These may

[1] Image irradiance is the light flux per unit area falling on the image, i.e., incident flux density. Scene radiance is the light flux per unit projected area per unit solid angle emitted from the scene, i.e., emitted flux density per unit solid angle.

be imposed by boundary conditions, by restrictions on the type of surface to be recovered, or by a combination of the two. For some images, when we can determine important features (such as the fact that an edge is an occlusion boundary caused by a surface turning smoothly away from the viewing direction), we can use boundary conditions to constrain the solution; in large portions of the image, however, we can say something only about the type of surface we would like to recover. To date surface smoothness is the weakest additional assumption that has allowed surface shape to be recovered. Smoothness normally signifies that the surface is continuous and that it is once or twice differentiable. Smoothness, as the additional assumption, has had to play the role of propagator of boundary conditions and selector of the surface to be recovered. Is smoothness capable of these tasks in general or is its usefulness limited to special cases?

In the first part of this paper we describe the various formulations that have employed smoothness, including a relaxation procedure of our own that resembles its counterpart in engineering; we then present results of our experiments with these iterative procedures. Assessing the usefulness of smoothness in this context, we conjecture as to its utility in other shape-from-shading formulations.

Not all authors have used smoothness as their additional constraint; some have employed assumptions about surface shape instead. The assumption that the surface is locally spherical, i.e., that its curvature is independent of direction, is strong enough to allow but a single interpretation for the surface orientation, and at the same time, it is also one that enables recovery of the surface orientation by purely local computation [8]. In addition, this shape constraint eliminates the need to know such parameters as illuminant direction and surface albedo.[2] Assumptions about shape are being traded for assumptions about reflectance behavior. Can we formulate the shape-from-shading problem without having to know the details of the surface reflectance and without making any assumptions about the shape of the surface we wish to recover?

In the new formulation presented in the second part of the paper, we assume that scene materials scatter light isotropically. We make no assumptions about surface shape and we do not need to know the parameters specifying illuminant direction, illuminant strength, and surface albedo. Our assumptions are about the properties of reflection in the world; these alone are sufficient to relate surface orientation to image irradiance. In situations in which the assumption of isotropic scattering is invalid, the formulation provides some interesting parallels to human vision.

2 ITERATIVE FORMULATIONS FOR SURFACE RECOVERY

The image irradiance equation as presented by Horn [2], is

$$I(x,y) = R(p,q) \quad ,$$

where $I(x,y)$ is the image irradiance as a function of the image coordinates x and y, and $R(p,q)$ is the surface radiance as a function of p and q, the derivatives of depth with respect

[2]Surface albedo is the material reflectance, i.e., the ratio of scene radiance to scene irradiance.

to the image coordinates. To derive this equation orthographic projection is assumed; while orthographic projection is inadequate to describe image formation it is a good approximation when the scene objects are small compared to the viewing distance. In the shape-from-shading approach, it is generally assumed that $R(p,q)$ is known for all p and q (that is, the reflectance map is specified). The iterative approach applies this equation on a pixel-by-pixel basis, that is, for pixel (i,j)

$$I_{i,j} = R(p_{i,j}, q_{i,j}) \quad ,$$

where $I_{i,j}$ is the image irradiance for (i,j)th pixel and $p_{i,j}$, $q_{i,j}$ is the surface orientation of the surface patch that is imaged at pixel (i,j). For convenience we use the notation

$$R_{i,j} \equiv R(p_{i,j}, q_{i,j}) \quad .$$

If, at some stage of the iterative procedure, we have assigned particular $p_{i,j}, q_{i,j}$ as the surface orientation of the (i,j)th pixel, then the residual expression

$$\xi_{i,j}{}^R = (I_{i,j} - R_{i,j})^2$$

specifies the error caused by our assignment of surface orientation.[3] If this were our only constraint, we could select $p_{i,j}, q_{i,j}$ so that $\xi_{i,j}{}^R = 0$. This would guarantee that the image irradiance equation is satisfied pixel by pixel, but, because there are infinitely many solutions, we need further constraints to reduce the number of possible solutions.

Smoothness is usually introduced by specifying a relationship that we would like to have hold between the surface orientation of the (i,j)th pixel and its neighbors. The various iterative approaches [5-7] differ in the way this relationship is specified. Of course, at a particular stage of the iterative process this relationship between a pixel and its neighbors will not be exact. Once again we can specify a residual equation for the error in the smoothness relation.

$$\xi_{i,j}{}^S = [f(p_{i,j}, q_{i,j}, p_{i-1,j}, q_{i-1,j}, p_{i+1,j}, q_{i+1,j}, p_{i,j-1}, q_{i,j-1}, p_{i,j+1}, q_{i,j+1}, \ldots)]^2 \quad ,$$

where f is the relationship between the surface orientation at (i,j) and its neighbors. An example of the type of relationship is the difference between the surface orientation of pixel (i,j) and the mean value of the surface orientations of its four-neighbors.

We have two constraints that need to be satisfied simultaneously, — one from image irradiance and one from surface smoothness. At each stage of the iterative process, the total residual error for pixel (i,j) can be described by

$$\xi_{i,j} = \lambda \xi_{i,j}{}^R + \xi_{i,j}{}^S \quad ,$$

where λ is a weighting factor that can adjust the influence of the error in image irradiance to the error in smoothness.[4] For the image, the total residual error is

$$\xi = \sum_{i,j} \xi_{i,j} \quad .$$

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[3]The form of the error need not be quadratic — the goals of such a choice include simple final expressions.

[4]Since the error in image irradiance is not necessarily commensurate with that in surface smoothness, some form of normalization is required.

The allocation of surface orientations to all pixels should minimize this total error, that is,

$$\frac{\partial \xi}{\partial p_{i,j}} = 0 \quad ,$$

$$\frac{\partial \xi}{\partial q_{i,j}} = 0 \quad .$$

Differentiating ξ with respect to $p_{i,j}$ and also with respect to $q_{i,j}$ gives two equations for each pixel in the image. While complicated forms of the relationship between $p_{i,j}, q_{i,j}$ and their neighboring pixels will generally occur, we choose our smoothness relation so that we can arrange the equations in open form

$$p_{i,j} = F_1(p_{i,j}, q_{i,j}, \textit{and p's and q's of neighboring pixels}) \quad ,$$

$$q_{i,j} = F_2(p_{i,j}, q_{i,j}, \textit{and p's and q's of neighboring pixels}) \quad ,$$

where F_1, and F_2 are functions.

We therefore have an iterative scheme that, given some initial solution, we improve by reducing the residual error in image irradiance and surface smoothness. We need to ask the following questions of such a scheme: Under what conditions will it converge to a solution? Is that solution unique? Does smoothness, as defined by our relation, give us the type of surface we want?

3 SURFACE ORIENTATION

There are many equivalent parameterizations of surface orientation. Mentioned previously were the parameters p and q, the derivatives of depth with respect to image coordinates. Some authors prefer using slant and tilt to specify surface orientation. Slant is the angle between the surface normal and the viewing direction, while tilt is the angle between the image x axis and the projection of the surface normal onto the image plane. Other parameterizations [7] have been used when particular properties of the parameterization are to be exploited. The parameters we use are l and m:[5]

$$l = \sin\sigma\cos\tau \quad ,$$
$$m = \sin\sigma\sin\tau \quad ,$$

where σ is the surface slant and τ its tilt, l is the component of the surface normal in the direction of the x axis, and m is the component in the y direction. We select this particular parameterization, as l and m are bounded

$$0 \leq l^2 + m^2 \leq 1 \quad .$$

For surfaces that we can see,

$$0 \leq \sigma < \frac{\pi}{2} \quad ,$$

[5] l and m are related to p and q, $\quad l = \frac{-p}{\sqrt{1+p^2+q^2}} \quad , m = \frac{-q}{\sqrt{1+p^2+q^2}} \quad .$

$$0 \leq \tau < 2\pi \quad .$$

Consequently l and m specify the surface normal of an imaged surface without ambiguity.

4 FORMULATIONS USED FOR SURFACE RECOVERY

To explore the issues of convergence, propagation of boundary conditions, and the type of surface promoted by smoothness, we formulate the problem in two ways: one that parallells the technique previously described and, alternatively, one that resembles the relaxation method used to solve structural engineering problems.

The function for scene radiance, used to create synthetic images for the experiment and employed by the shape recovery algorithms, is

$$R(l,m) = 0.1569(\frac{1+\sqrt{1-l^2-m^2}}{2}) + Max[0.4437\sqrt{1-l^2-m^2} + 0.3137l + 0.3137m, 0].$$

This function is appropriate for a scene that exhibits Lambertian reflectance and is illuminated by both a collimated source and a uniform hemispherical source. This illumination was selected because it is typical of the illumination of outdoor scenes. The particular numerical constants specify the light direction and intensity, and the surface albedo.

The first formulation is similar to that described previously; we shall call this the 'conventional' formulation. From the image irradiance equation we have the error term

$$\xi_{i,j}{}^R = (I_{i,j} - R_{i,j})^2 \quad .$$

The smoothness constraint is the requirement that $l_{i,j}$ be the average of its four-neighbors, and that $m_{i,j}$ be the average of its four-neighbors. The error term for smoothness is

$$\begin{aligned} \xi_{i,j}{}^S = \quad & (l_{i,j} - \frac{l_{i-1,j} + l_{i+1,j} + l_{i,j-1} + l_{i,j+1}}{4})^2 \\ & + (m_{i,j} - \frac{m_{i-1,j} + m_{i+1,j} + m_{i,j-1} + m_{i,j+1}}{4})^2 \quad . \end{aligned}$$

Note that this constraint is exact for a surface that is spherical.

Minimizing $\xi = \sum_{i,j} \lambda \xi_{i,j}{}^R + \xi_{i,j}{}^S$ by differentiating with respect to $l_{i,j}$, and with respect to $m_{i,j}$, and then setting each result equal to zero, we obtain the expressions

$$\begin{aligned} l_{i,j} = \quad & 0.4(l_{i-1,j} + l_{i+1,j} + l_{i,j-1} + l_{i,j+1}) - \\ & 0.1(l_{i-1,j-1} + l_{i+1,j+1} + l_{i-1,j+1} + l_{i+1,j-1}) - \\ & 0.05(l_{i-2,j} + l_{i+2,j} + l_{i,j-2} + l_{i,j+2}) + \\ & 0.8\lambda(I_{i,j} - R_{i,j}) \left.\frac{\partial R}{\partial l}\right|_{i,j} \quad , \\ m_{i,j} = \quad & 0.4(m_{i-1,j} + m_{i+1,j} + m_{i,j-1} + m_{i,j+1}) - \\ & 0.1(m_{i-1,j-1} + m_{i+1,j+1} + m_{i-1,j+1} + m_{i+1,j-1}) - \\ & 0.05(m_{i-2,j} + m_{i+2,j} + m_{i,j-2} + m_{i,j+2}) + \\ & 0.8\lambda(I_{i,j} - R_{i,j}) \left.\frac{\partial R}{\partial m}\right|_{i,j} \quad . \end{aligned}$$

We use these (together with the expression for $R(l,m)$) as our iterative scheme to improve on an intial solution.

The other formulation we use, the 'engineering' formulation, creates error terms from the image irradiance equation and the smoothness constraints, but does not combine these into one term.

$$\xi_{i,j}{}^R = (I_{i,j} - R_{i,j}) \quad ,$$

$$\xi_{i,j}{}^{S_1} = (l_{i,j} - \frac{l_{i-1,j} + l_{i+1,j} + l_{i,j-1} + l_{i,j+1}}{4}) \quad ,$$
$$\xi_{i,j}{}^{S_2} = (m_{i,j} - \frac{m_{i-1,j} + m_{i+1,j} + m_{i,j-1} + m_{i,j+1}}{4}) \quad .$$

We view the ξ's as residuals and apply the relaxation approach, i.e., reduction of the largest residuals. If $\xi_{i,j}{}^{S_1}$ or $\xi_{i,j}{}^{S_2}$ is selected for reduction we choose to reduce both, as each is independent of the other. When $\xi_{i,j}{}^R$ is chosen for reduction, we do the reduction in two stages — one stage altering $l_{i,j}$ and the other $m_{i,j}$. Of course we can scale the residuals, reduce them from, say, the image irradiance equation to a certain level before introducing smoothness, vary the amount of correction we apply, (e.g., we can overrelax) and the like. In fact, we can experiment with various relaxation approaches. In this formulation major changes in the relaxation scheme generally require minor programming changes.

5 EXPERIMENTAL RESULTS

The test image shown in Figure 1 is that of a hemisphere placed on a plane, i.e., a synthetic image generated by the reflectance function previously described. The collimated light source is at slant $\frac{\pi}{4}$ and tilt $\frac{\pi}{4}$ — which means that it is at the upper right as we view the image. We purposely avoided the case in which the collimated source is at the same position as the viewer, since the resulting symmetric reflectance map might bias the algoritbm to return a symmetric surface. A synthetic image of a sphere was selected as the test image because both the image irradiance equation and the smoothness relationship we use hold exactly.[6] The performance of the algorithm to recover the surface shape could be assessed without the complications involved in using inexact models for reflectance and smoothness.

We need initial solutions to start our iterative/relaxation procedures. We used four sets of initial conditions: (1) a plane perpendicular to the viewing direction; (2) a plane slanted $\frac{\pi}{4}$ to the viewing direction; (3) a cone with its axis along the viewing direction; (4) the correct solution perturbed by small random errors.

Previous work bas used boundary conditions to constrain the recovered surface. Investigating this approach, we constrained the surface in various ways: at the edge of the hemisphere, at a closed curve lying on the sphere's surface, or at individual points on the sphere's surface. We also used the algorithms without any boundary conditions whatsoever.

Since we wished to investigate the extent to which smoothness could propagate boundary conditions, we used various image quantizations, namely 16×16, 32×32, and 64×64.

[6]The smoothness relationship does not hold at the edge of the hemisphere where it joins the plane.

The findings can be characterized as follows:

- Both techniques — the engineering and the conventional method — gave essentially the same results.
- The engineering technique converged much faster than the conventional technique.
- Smoothness propagates boundary conditions by no more than a few pixels
- The initial solution largely predetermines the final one.

Figures 3-11 display examples of the results we achieved with the conventional iterative scheme; the engineering scheme gave essentially the same results. In each of these figures the top left picture shows the profile of the recovered surface (viewed from the bottom left corner) , while at the top right we find an image that is the sine of the surface slant, with black representing 0 and white 1. The bottom left is the cosine and the bottom right the sine of the surface tilt, with black representing -1, gray 0, and white +1. The results are presented in this manner so that the performance of the algorithms can be evaluated. The profile can on occasion appear more accurate than the individual surface orientations (as might be expected of an integration procedure); at other times, however, errors in the surface orientation (sometimes just from the image quantization) of highly slanted surfaces cause the integration routine that produces the surface shape for profiling to overstate the error. Figure 2 shows the results that should be obtained if the shape recovery algorithms recovered the surface exactly.

Figures 3-6 illustrate the effects of various boundary conditions. The errors at the edge of the sphere where it joins the plane are expected, as smoothness does not hold there. Each figure is the result of 320 iterations, this being five times the linear dimension of the picture used. The boundary condition at a point affects an area of approximately 10 pixels in radius. Only for Figure 6, where a random five percent of pixels were set to their correct values, is the surface shape recovered correctly. Smoothness as a propagator affects but a small area. Figures 4, 7, and 8 illustrate this point further. Here various image sizes are used. Observe that, as the image size increases, the boundary conditions diminish in their effect and the solution becomes progressively worse. Figures 4, 9, and 10 reveal the dominant influence of the initial solution. Figure 11 is included to show the effect of smoothness when $\lambda = 0$ — namely, when image irradiance does not affect the solution at all. This figure, obtained after 320 iterations, demonstrates what smoothness alone can achieve, even when the definition of smoothness is exact for the viewed scene (a sphere).

Smoothness is a poor selector of surface shape and a poor propagator of boundary information when it is used to tie the surface orientation of a particular surface point to those of its neighbors. Generally, in engineering, problems solved with relaxation techniques are formulations that relate a given property at one point to that same property at neighboring points by means of differential relations. It is the derivative that propagates boundary information and selects a particular solution to be recovered. We present such a formulation below in an attempt to relieve smoothness of its role as propagator and selector.

6 SURFACE RADIANCE AND ISOTROPIC SCATTERING

Our formulation of the relationship between image irradiance and scene radiance is

$$I(x,y) = R(l,m) \quad ,$$

where $I(x,y)$ is the image irradiance at image point x,y and $R(l,m)$ is the scene radiance for a surface normal we represent by l,m. R is a function of the components of the surface normal and they, in turn, are functions of image coordinates. $R(l,m)$ specifies the relationship between surface radiance and surface orientation, while $l(x,y)$ and $m(x,y)$ specify the relationship between surface orientation and image coordinates. $R(l,m)$ embodies knowledge of the nature of surface reflection, while $l(x,y)$ and $m(x,y)$ embody the surface shape.

To provide the additional constraints we need for relating surface orientation to image irradiance, we introduce constraints that relate properties of $R(l,m)$, — that is, constraints that specify the relationship between surface radiance and surface orientation. Such constraints are

$$(1-l^2)R_{ll} = (1-m^2)R_{mm} \quad ,$$

$$(R_{ll} - R_{mm})lm = (l^2 - m^2)R_{lm} \quad ,$$

where R_{ll} is the second partial derivative of R with respect to l, R_{mm} is the second partial derivative of R with respect to m, and R_{lm} is the second partial cross-derivative of R with respect to l and m.

These two partial differential equations embody the assumption of isotropic scattering (Lambertian reflectance). For isotropic scattering $R(l,m)$ has the form

$$R(l,m) = al + bm + c\sqrt{1-l^2-m^2} + d \quad ,$$

where $a, b, c,$ and d are constants, their values depending on illumination conditions and surface albedo. Note that l,m, and $\sqrt{1-l^2-m^2}$ are the components of the unit surface normal in the directions x,y, and depth. $R(l,m)$ can be viewed as the dot product of the surface normal vector $(l, m, \sqrt{1-l^2-m^2})$ and a vector (a,b,c) denoting illumination conditions. As the value of a dot product is rotationally independent of the coordinate system, the scene radiance is independent of the viewing direction — which is the definition of isotropic scattering.

It is easily seen that $R(l,m) = al + bm + c\sqrt{1-l^2-m^2} + d$ satisfies the pair of partial differential equations given above. In the appendix we show that $R(l,m) = al + bm + c\sqrt{1-l^2-m^2} + d$ is the solution of the pair of partial differential equations. These partial differential equations are an alternative definition of isotropic scattering.

It is worthy of note that $R(l,m) = al + bm + c\sqrt{1-l^2-m^2} + d$ includes radiance functions for multiple and extended illumination sources, including that for a hemispherical uniform source such as the sky. The assumption of isotropic scattering restricts the class of material surfaces being considered, not the illumination conditions.

7 EQUATIONS RELATING SURFACE ORIENTATION TO IMAGE IRRADIANCE

Differentiating

$$I(x,y) = R(l,m)$$

with respect to x and y, we obtain

$$I_x = R_l l_x + R_m m_x \quad ,$$

$$I_y = R_l l_y + R_m m_y \quad ,$$

$$I_{xx} = R_{ll} {l_x}^2 + R_{mm} {m_x}^2 + 2R_{lm} l_x m_x + R_l l_{xx} + R_m m_{xx} \quad ,$$

$$I_{yy} = R_{ll} {l_y}^2 + R_{mm} {m_y}^2 + 2R_{lm} l_y m_y + R_l l_{yy} + R_m m_{yy} \quad ,$$

$$I_{xy} = R_{ll} l_x l_y + R_{mm} m_x m_y + R_{lm}(l_x m_y + l_y m_x) + R_l l_{xy} + R_m m_{xy} \quad ,$$

where subscripted variables denote partial differentation with respect to the subscript(s).

From the constraints for isotropic scattering, we derive the relationships

$$R_{ll} = \frac{1 - m^2}{lm} R_{lm} \quad ,$$
$$R_{mm} = \frac{1 - l^2}{lm} R_{lm} \quad .$$

Substituting these relationships for R_{ll} and R_{mm} in the expressions for I_{xx},I_{yy},and I_{xy}, we obtain

$$[{l_x}^2(\frac{1-m^2}{lm}) + {m_x}^2(\frac{1-l^2}{lm}) + 2l_x m_x]R_{lm} = I_{xx} - R_l l_{xx} - R_m m_{xx} \quad ,$$
$$[{l_y}^2(\frac{1-m^2}{lm}) + {m_y}^2(\frac{1-l^2}{lm}) + 2l_y m_y]R_{lm} = I_{yy} - R_l l_{yy} - R_m m_{yy} \quad ,$$
$$[l_x l_y(\frac{1-m^2}{lm}) + m_x m_y(\frac{1-l^2}{lm}) + l_x m_y + l_y m_x]R_{lm} = I_{xy} - R_l l_{xy} - R_m m_{xy} \quad .$$

By removing R_{lm} and substituting the expressions for R_l and R_m, defined by the expressions for I_x and I_y, we produce two partial differential equations relating surface orientation to image irradiance:

$$\alpha\theta l_{xx} + \beta\theta m_{xx} - \alpha\gamma l_{xy} - \beta\gamma m_{xy} = \chi\theta I_{xx} - \chi\gamma I_{xy} \quad ,$$
$$\alpha\theta l_{yy} + \beta\theta m_{yy} - \alpha\delta l_{xy} - \beta\delta m_{xy} = \chi\theta I_{yy} - \chi\delta I_{xy} \quad ,$$

where

$$\alpha = I_x m_y - I_y m_x \quad ,$$
$$\beta = I_y l_x - I_x l_y \quad ,$$
$$\gamma = {l_x}^2(1 - m^2) + {m_x}^2(1 - l^2) + 2l_x m_x lm \quad ,$$
$$\delta = {l_y}^2(1 - m^2) + {m_y}^2(1 - l^2) + 2l_y m_y lm \quad ,$$
$$\theta = l_x l_y(1 - m^2) + m_x m_y(1 - l^2) + (l_x m_y + l_y m_x)lm \quad ,$$
$$\chi = l_x m_y - l_y m_x \quad .$$

These equations relate surface orientation to image irradiance by parameter-free expressions. They involve the derivatives of image irradiance, but not the image irradiance itself — an important feature if we conjecture such a model for the human visual system.

8 RECOVERY OF SURFACE SHAPE — A SPECIAL CASE: A SPHERICAL SURFACE

It is difficult to solve the equations relating surface orientation to image irradiance, and thus to recover surface shape from observed image irradiance. Two types of approaches are possible. The two differential equations can be integrated in a step-by-step manner or, given some initial solution, a relaxation procedure may be employed. The difficulties that arise are two-fold, numerical errors and multiple solutions.

Solutions of the equation $\chi = 0$ (the developable surfaces, e.g., a cylinder) are also solutions of the equations relating surface orientation to image irradiance. If the images intensities were known in analytic form then analytic solution of the equations could employ boundary conditions to select the appropiate solution. However since the analytic form for the image intensities is unknown, numerical procedures must be employed. Numerical procedures to integrate the equations inevidently introduces small errors. Instability of the numerical scheme seems responsible for such errors eventually dominating the recovered solution.

The alternative, a relaxation procedure to solve the equations, has its own difficulties. The difficulties experienced in the shape-from-shading methods discussed in the first part of this paper dictate caution. The importance of a good initial solution for a relaxation method cannot be overemphasized. Simplfying the two partial differential equations (using additional assumptions) provides a method for obtaining an good initial solution.

The spherical approximation assumes that we are on a spherical surface. This implies $l_y = 0, m_x = 0, l_x = m_y,$ — namely, constant curvature independent of direction. For this case the partial differential equations become relationships between the second derivatives of image irradiance and the components of the surface normal:

$$\frac{1-m^2}{lm} = \frac{I_{xx}}{I_{xy}} \quad ,$$

$$\frac{1-l^2}{lm} = \frac{I_{yy}}{I_{xy}} \quad .$$

These results for the spherical approximation are equivalent to those Pentland was able to obtain [8] through local analysis of the surface. In addition to providing a mechanism for obtaining an initial solution for a relaxation-style algorithm, their direct application estimates the surface orientation by local computation [8].

We are actively engaged in tbe development of a relaxation procedure to transform the initial solution (given by the spherical approximation) into a solution the satifies the full equations.

9 THE INFLUENCE OF BELIEF ON THE PERFORMANCE OF A VISUAL SYSTEM

The constraints derived for isotropic scattering do not have to be true embodiments of the physical laws of nature; rather, they can represent the beliefs a visual system possesses regarding those laws. In circumstances in which such beliefs do not hold, the visual system will err in predicting the world's true nature. Of course, if the model is not a good approximation of the physical laws of nature, the visual system embodying it is useless.

The two constraints specifying isotropic scattering,

$$(1-l^2)R_{ll} = (1-m^2)R_{mm} \quad ,$$
$$(R_{ll} - R_{mm})lm = (l^2 - m^2)R_{lm} \quad ,$$

obviously both hold when the scattering is isotropic, but what is the situation for other forms of scattering?

The images produced by a scanning electron microscope constitute an intriguing situation. The appropiate expression for scene radiance [7] is

$$R(l,m) = a(1 + \frac{1}{\sqrt{1-l^2-m^2}}) \quad ,$$

where a is a constant. This expression is quite unlike those for natural scenery, yet the human visual system 'sees' an image. Note that the second constraint for the isotropic scattering case is satisfied by this radiance function, but not the first. The second constraint is about surface tilt, as $\frac{lm}{l^2-m^2} = \frac{\tan 2\tau}{2}$, where τ is the surface tilt; the first constraint introduces slant. In using the equations relating surface orientation to image irradiance to recover surface orientation, one might expect them to predict tilt correctly for surfaces in electron microscope images, but to err in predicting slant.

For other forms of the scene radiance expressions, neither constraint holds. Specular reflectance has been approximated [2] by

$$R(l,m) = \quad a[b(1-l^2-m^2) + cl\sqrt{1-l^2-m^2} + dm\sqrt{1-l^2-m^2}]^n \quad ,$$

where n is a constant, usually having a value between 1 and 10 that determines the 'sharpness' of the specular peak.

For the maria of the moon, the form of scene radiance normally used [2] is

$$R(l,m) = \frac{a(bl+cm)}{\sqrt{1-l^2-m^2}} \quad .$$

The constants a, b, c, and d in the above expressions are associated with the strength and position of the light source, as well as with the surface albedo.

The constraints do not hold in either of the preceding cases. We would expect a visual system embodying them to make errors under these circumstances. Nevertheless this should not induce us to immediately begin searching for new visual beliefs. After all the human visual system is imperfect under conditions of specular reflection; moreover, people observed the moon throughout history without concluding that it was spherical.

If these constraints are incorporated in the human visual system, the predictions based on them — i.e., when the visual system will return ostensibly 'correct' and 'incorrect' information — could be tested by psychophysical experiments. Such predictions together with their verification or refutation are being investigated.

10 CONCLUSION

The shape-from-shading task (recovering surface orientation from image irradiance), has meant finding a solution to the image irradiance equation. This formulation requires that the characteristics of the scene illumination and the surface material be known. While these requirements are difficult to satisify,knowing them makes it possible to apply the image irradiance equation to any scene material for which the scene radiance function is known. Such application, however, is not without difficulty, appropiate boundary conditions are needed and the effect of image noise is uncertain.

To recover surface orientation, relaxation-style algorithms based on the image irradiance equation employ additional constraints. These constraints, which are needed to supplement the underdetermined image irradiance equation, capture the concept of smoothness. While smoothness superficially determines the relationship between image irradiance and surface orientation, it is too weak a concept to propagate boundary conditions and thus equally ineffectual as a means of recovering the required solution.

In presenting a new formulation for the shape-from-shading task, we have traded the need to know the explicit form of the scene radiance function for the assumption that material scatters light isotropically. This model is applicable to natural scenery without additional assumptions about illumination conditions or the albedo of the surface material. The model also demonstrates some competence even when the scattering is not isotropic. Such a model poses the question: does the human visual system embody a particular belief about the laws of scattering that it applies even when these laws are inexact?

Effective numerical procedures based on this new formulation of the shape-from- shading task remain unknown and, are subjects for further development.

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APPENDIX

We show that the solution of the system of partial differential equations,

$$(1-l^2)R_{ll} = (1-m^2)R_{mm} \quad ,$$
$$(R_{ll} - R_{mm})lm = (l^2 - m^2)R_{lm} \quad ,$$

where R_{ll} is the second partial derivative of R with respect to l, R_{mm} is the second partial derivative of R with respect to m, and R_{lm} is the second partial cross derivative of R with respect to l and m, is

$$R(l,m) = al + bm + c\sqrt{1-l^2-m^2} + d \quad ,$$

where a, b, c, and d are constants.

Proof: Rearranging

$$(1-l^2)R_{ll} = (1-m^2)R_{mm} \quad ,$$
$$(R_{ll} - R_{mm})lm = (l^2 - m^2)R_{lm} \quad ,$$

we obtain

$$R_{lm} = \frac{lm}{1-m^2}R_{ll} \quad ,$$
$$R_{lm} = \frac{lm}{1-l^2}R_{mm} \quad .$$

Integrating $R_{lm} = \frac{lm}{1-m^2}R_{ll}$ with respect to l, that is,

$$\int R_{lm}\, dl = \frac{m}{1-m^2}\int lR_{ll}\, dl \quad ,$$

gives

$$R_m = \frac{m}{1-m^2}(lR_l - R) + F_1(m) \quad ,$$

where $F_1(m)$ is an arbitary function of m. Similarly, integrating $R_{lm} = \frac{lm}{1-l^2}R_{mm}$ with respect to m gives

$$R_l = \frac{l}{1-l^2}(mR_m - R) + F_2(l) \quad ,$$

where $F_2(l)$ is an arbitary function of l. Rearranging these two equations, we get a system of two first order partial differential equations

$$(1-l^2-m^2)R_l + lR = lmF_3(m) + (1-m^2)F_4(l) \quad ,$$
$$(1-l^2-m^2)R_m + mR = lmF_4(l) + (1-l^2)F_3(m) \quad ,$$

where $F_3(m) = (1-m^2)F_1(m)$, and $F_4(l) = (1-l^2)F_2(l)$. Multiplying both equations by the integrating factor $(1-l^2-m^2)^{-\frac{3}{2}}$, we obtain

$$\frac{\partial}{\partial l}[(1-l^2-m^2)^{-\frac{1}{2}}R] = (1-l^2-m^2)^{-\frac{3}{2}}[lmF_3(m) + (1-m^2)F_4(l)] \quad ,$$
$$\frac{\partial}{\partial m}[(1-l^2-m^2)^{-\frac{1}{2}}R] = (1-l^2-m^2)^{-\frac{3}{2}}[lmF_4(l) + (1-l^2)F_3(m)] \quad .$$

Before carrying out the integration, we can find the form of $F_3(m)$, and $F_4(l)$ by differentiating the first equation with respect to m and the second with respect to l:

$$\frac{\partial^2}{\partial l \partial m}[(1-l^2-m^2)^{-\frac{1}{2}}R] = (1-l^2-m^2)^{-\frac{5}{2}}[l(1-l^2+2m^2)F_3(m) + m(1-m^2+2l^2)F_4(l) + lm(1-l^2-m^2)F_3'(m)] \quad ,$$

$$\frac{\partial^2}{\partial l \partial m}[(1-l^2-m^2)^{-\frac{1}{2}}R] = (1-l^2-m^2)^{-\frac{5}{2}}[l(1-l^2+2m^2)F_3(m) + m(1-m^2+2l^2)F_4(l) + lm(1-l^2-m^2)F_4'(l)] \quad ,$$

where $F'(k)$ indicates differentiation with respect to the independent variable k. Hence,

$$F_3'(m) = F_4'(l) \quad .$$

$F_3(m)$ is a function of m and $F_4(l)$ is a function of l; this implies that

$$F_3'(m) = d \quad ,$$
$$F_4'(l) = d \quad ,$$

where d is a constant. Therefore,

$$F_3(m) = dm + b \quad ,$$
$$F_4(l) = dl + a \quad ,$$

where a, and b are constants. Returning to the integration step, we now have the expressions

$$\frac{\partial}{\partial l}[(1-l^2-m^2)^{-\frac{1}{2}}R] = (1-l^2-m^2)^{-\frac{3}{2}}[l(bm+d) + a(1-m^2)] \quad ,$$
$$\frac{\partial}{\partial m}[(1-l^2-m^2)^{-\frac{1}{2}}R] = (1-l^2-m^2)^{-\frac{3}{2}}[m(al+d) + b(1-l^2)] \quad .$$

Integrating the first equation with respect to l and the second with respect to m, we obtain

$$(1-l^2-m^2)^{-\frac{1}{2}}R = (bm+d)(1-l^2-m^2)^{-\frac{1}{2}} + al(1-l^2-m^2)^{-\frac{1}{2}} + F_5(m) \quad ,$$
$$(1-l^2-m^2)^{-\frac{1}{2}}R = (al+d)(1-l^2-m^2)^{-\frac{1}{2}} + bm(1-l^2-m^2)^{-\frac{1}{2}} + F_6(l) \quad ,$$

where $F_5(m)$, and $F_6(l)$ are arbitary functions of m and l, respectively. We have two expressions for R:

$$R = al + bm + F_5(m)(1-l^2-m^2)^{\frac{1}{2}} + d \quad ,$$
$$R = al + bm + F_6(l)(1-l^2-m^2)^{\frac{1}{2}} + d \quad ,$$

which are compatible if

$$F_5(m) = F_6(l) = c \quad ,$$

where c is a constant. The solution for R is

$$R = al + bm + c\sqrt{1-l^2-m^2} + d \quad .$$

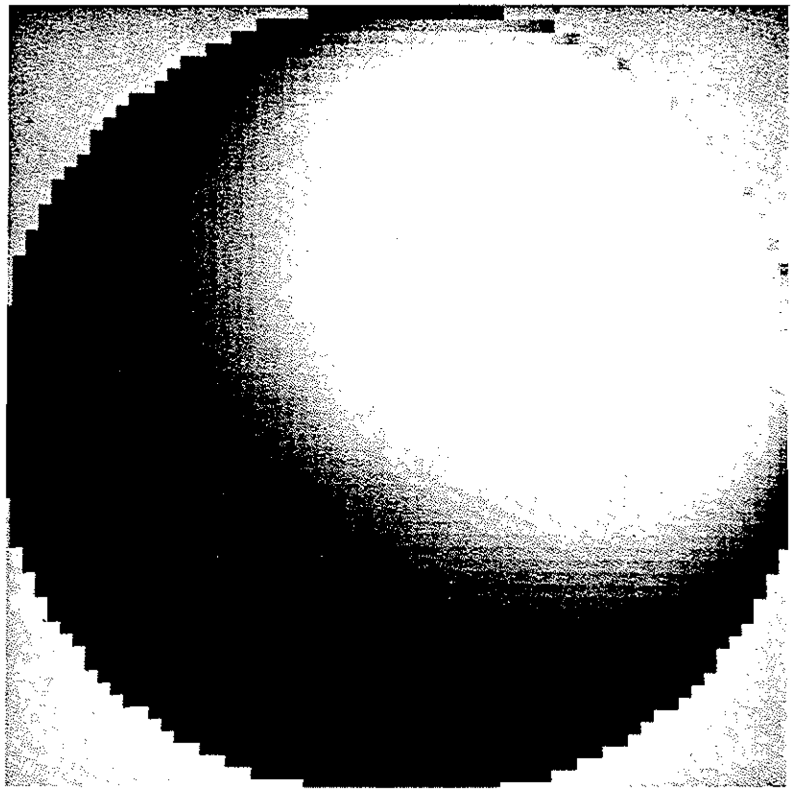

Figure 1 Original Image.

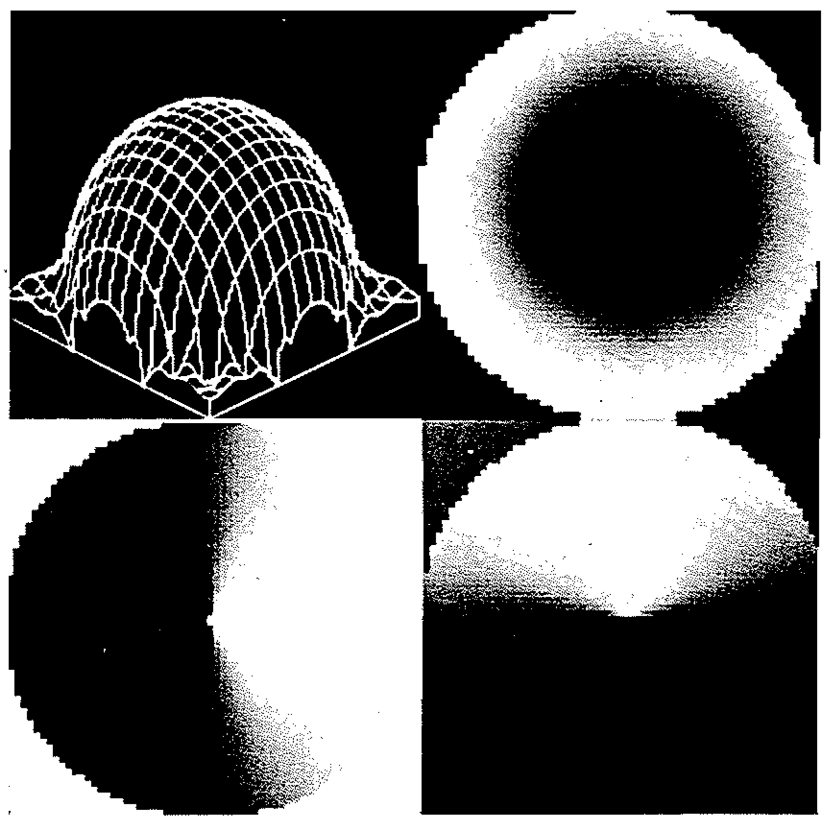

Figure 2 'Ideal' Result. Top left - profile of recovered surface; top right - sine slant, black≡0, white≡1; bottom left - cosine tilt, black≡-1, gray≡0, white≡+1; bottom right - sine tilt.

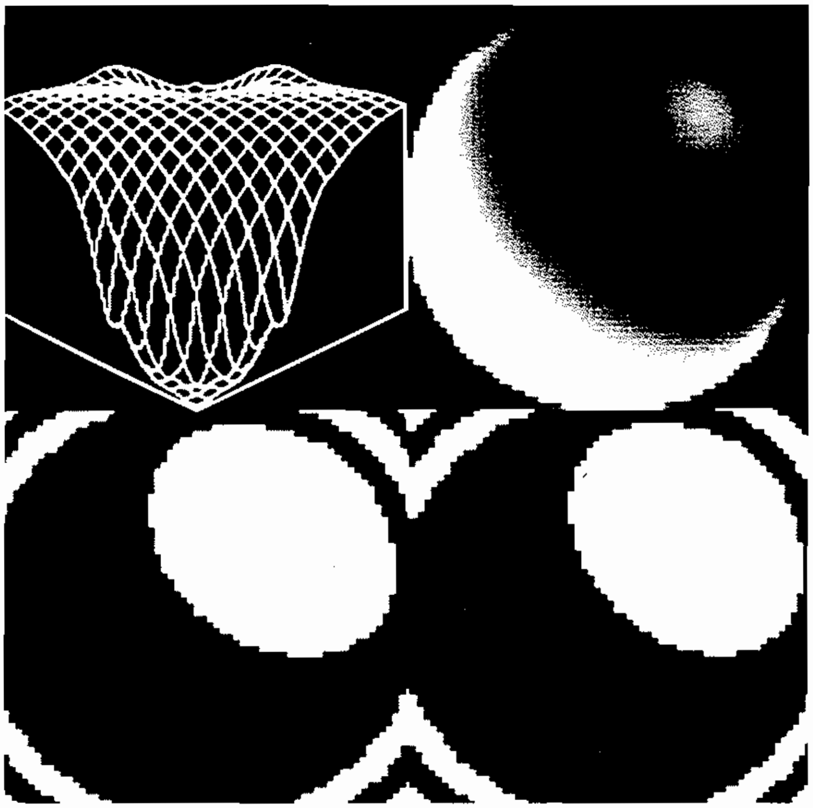

Figure 3. No boundary conditions; planar initial solution perpendicular to viewing direction; image quantization 64 × 64.

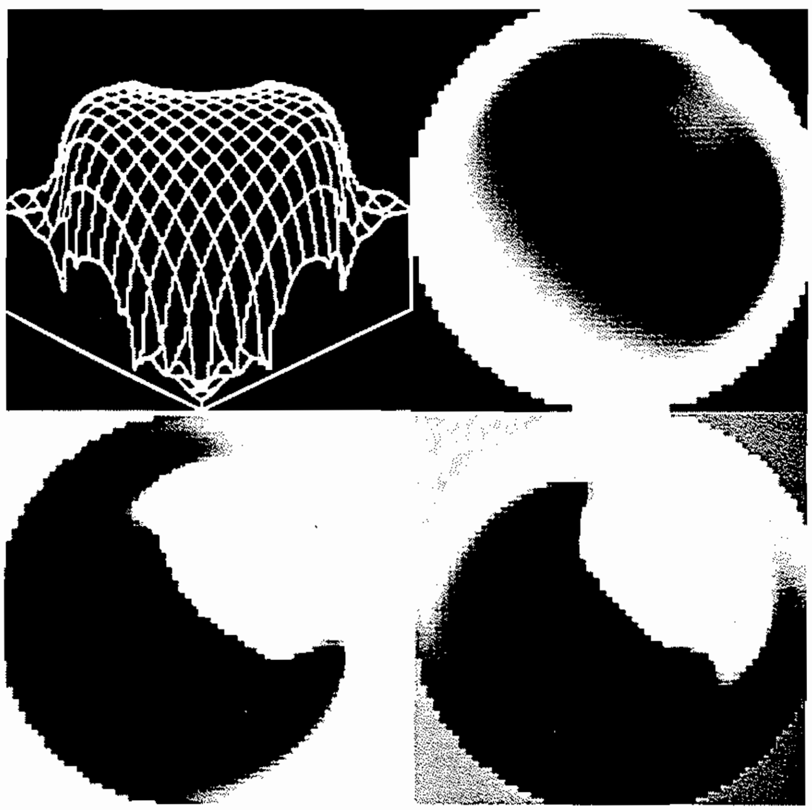

Figure 4. Boundary at edge of sphere given; planar initial solution perpendicular to viewing direction; image quantization 64 × 64.

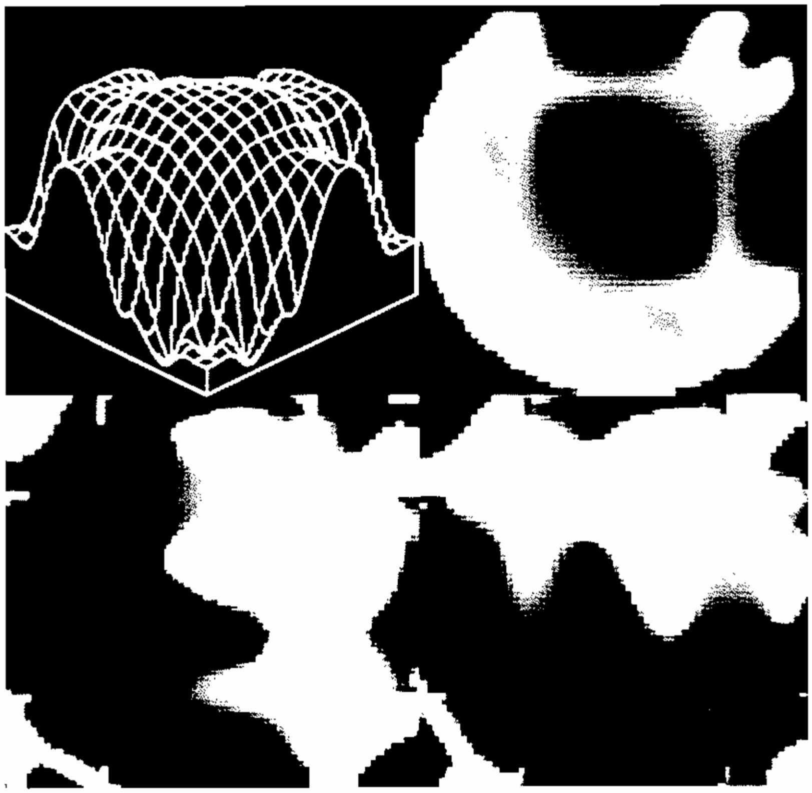

Figure 5. Boundary condition curve on sphere's surface (square shape); planar initial solution perpendicular to viewing direction; image quantization 64 × 64.

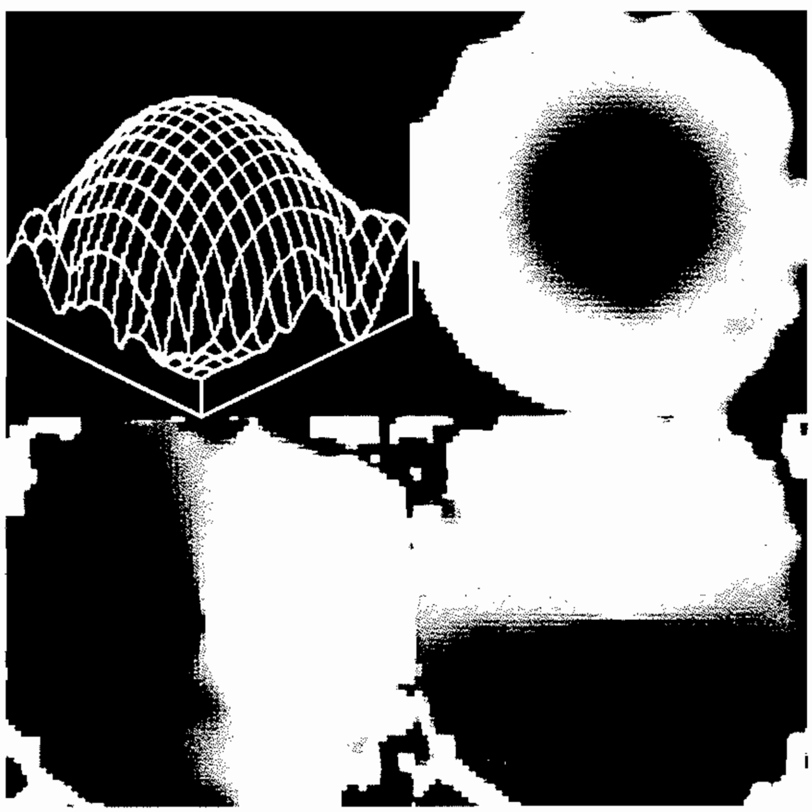

Figure 6. Random five percent of points fixed; planar initial solution perpendicular to viewing direction; image quantization 64 × 64.

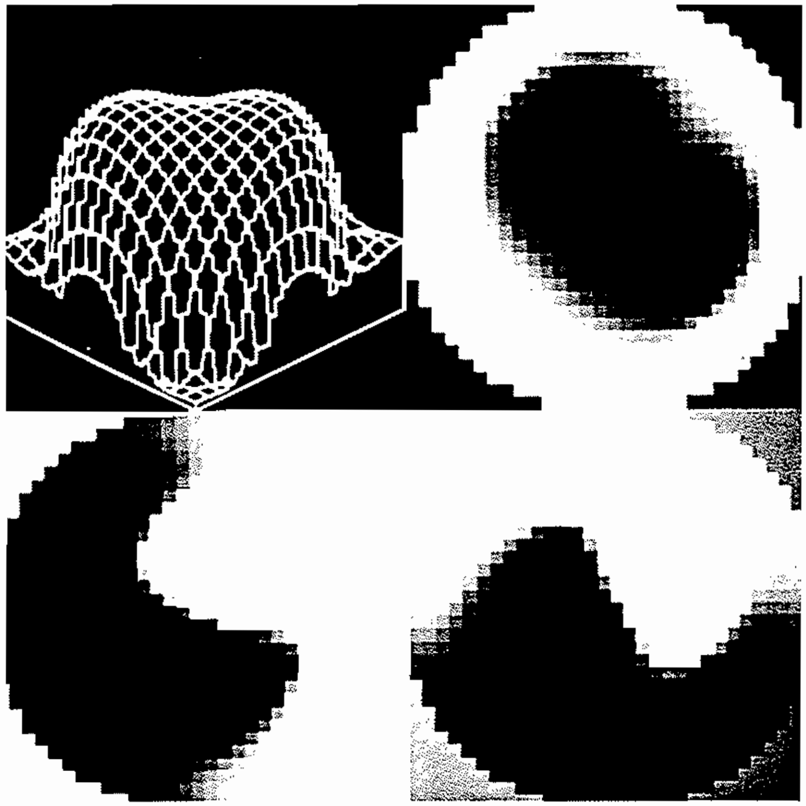

Figure 7. Boundary at edge of sphere given; planar initial solution perpendicular to viewing direction; image quantization 32 × 32.

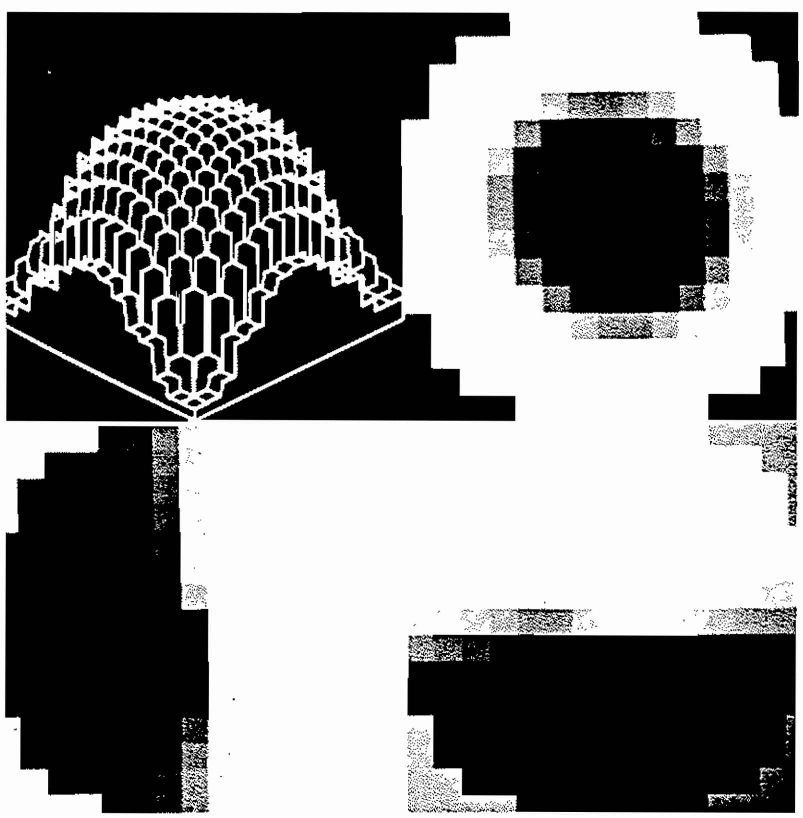

Figure 8. Boundary at edge of sphere given; planar initial solution perpendicular to viewing direction; image quantization 16 × 16.

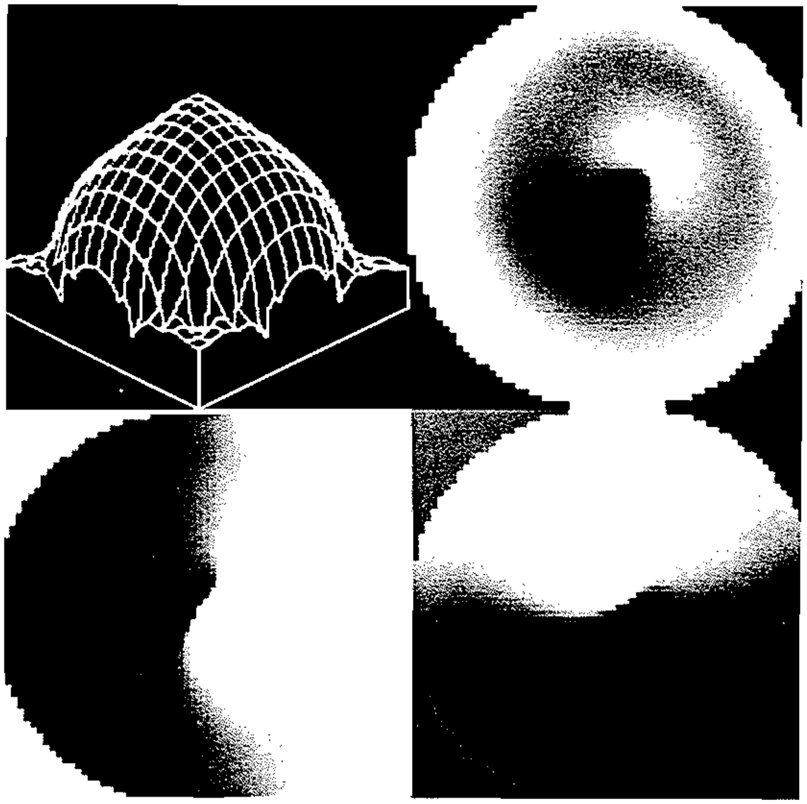

Figure 9. Boundary at edge of sphere given; conical initial solution; image quantization 64 × 64.

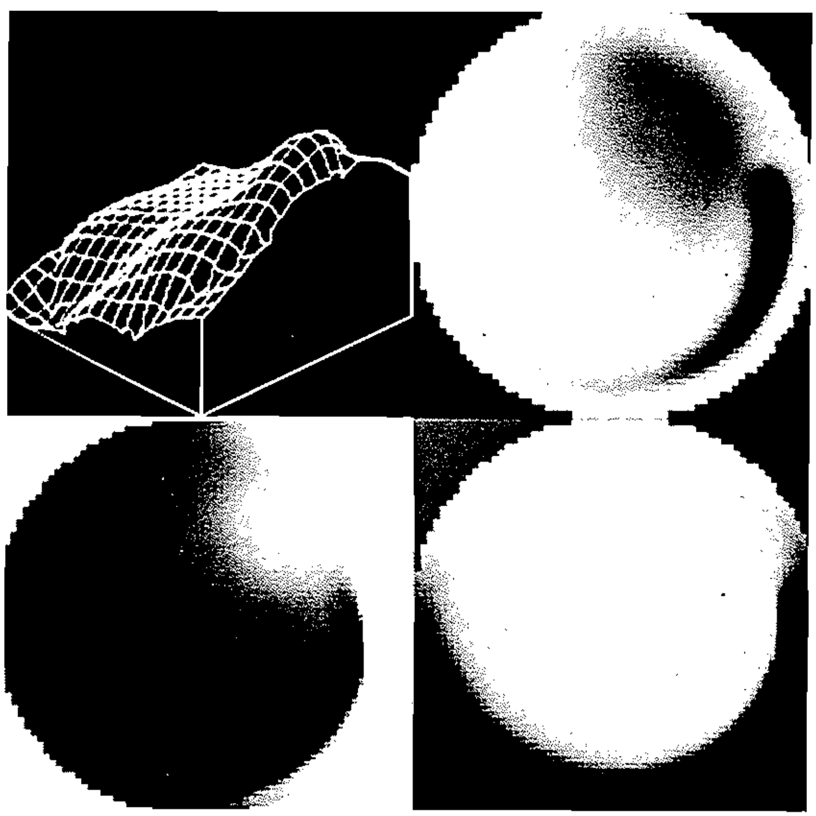

Figure 10. Boundary at edge of sphere given; planar initial solution slanted $\frac{\pi}{4}$ to viewing direction; image quantization 64 × 64.

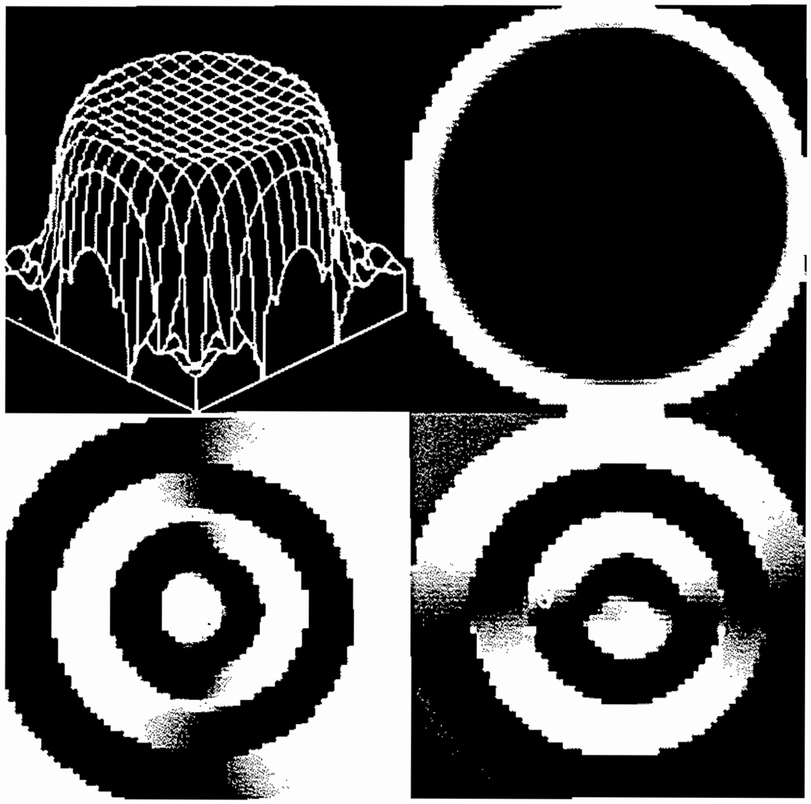

Figure 11. Smoothness constraint only. Boundary at edge of sphere given; planar initial solution perpendicular to viewing direction; image quantization 64 × 64.